

ON THE DEFINING RELATIONS FOR GENERALIZED q -SCHUR ALGEBRAS

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ABSTRACT. We show that the defining relations needed to describe a generalized q -Schur algebra as a quotient of a quantized enveloping algebra are determined completely by the defining ideal of a certain finite affine variety, the points of which correspond bijectively to the set of weights. This explains, unifies, and extends previous results.

INTRODUCTION

Consider a reductive Lie algebra \mathfrak{g} and an associated quantized enveloping algebra \mathbf{U} (over $\mathbb{Q}(v)$, v an indeterminate) determined by a given root datum $(X, \{\alpha_i\}, Y, \{h_i\})$ of finite type. (See §1 for basic notation and standard terminology.) In some sense \mathbf{U} is a deformation of the universal enveloping algebra $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ (taken over \mathbb{Q}). A generalized q -Schur algebra $\mathbf{S}(\pi)$ determined by a finite saturated set $\pi \subset X^+$ is the quotient \mathbf{U}/Λ where Λ consists of all $u \in \mathbf{U}$ acting as zero on all finite dimensional \mathbf{U} -modules admitting a weight space decomposition indexed by $W\pi$ (W is the Weyl group). Thus one truncates the category of \mathbf{U} -modules to the full subcategory $\mathcal{C}[\pi]$ consisting of those modules admitting such a weight space decomposition, and the module category of the finite dimensional algebra $\mathbf{S}(\pi)$ is precisely the category $\mathcal{C}[\pi]$.

We consider the general problem of describing in some explicit way the extra relations needed to define $\mathbf{S}(\pi)$ as a quotient of \mathbf{U} ; that is, we seek to find a reasonable set of generators of the ideal Λ in terms of the usual generators of \mathbf{U} . Then $\mathbf{S}(\pi)$ is the algebra given by the usual generators of \mathbf{U} with its usual defining relations along with the extra relations generating Λ .

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We also consider the analogous question in the limit at $v = 1$, where $\mathbf{S}(\pi)$ is replaced by a generalized Schur algebra $S(\pi)$ and \mathbf{U} is replaced by \mathfrak{U} . The generalized Schur algebra $S(\pi)$ was introduced by Donkin [3] as the quotient \mathfrak{U}/Λ where again Λ consists of all $u \in \mathfrak{U}$ which act as zero on all finite dimensional \mathfrak{U} -modules admitting a weight space decomposition indexed by the set $W\pi$.

Our main results are Theorems 4.13 and 5.3 and Corollaries 4.17 and 5.4. The theorems state that the extra relations may be found simply by locating a set of generators for the vanishing ideal of a certain discrete affine variety corresponding to the set $W\pi$ of weights (W is the Weyl group), and the corollaries give explicit formulas for a set of generators of that ideal. More precisely, we choose an arbitrary \mathbb{Z} -basis H_1, \dots, H_n of the abelian group Y . In the classical case, the discrete set of points is

$$\{(\langle H_1, \lambda \rangle, \dots, \langle H_n, \lambda \rangle) \mid \lambda \in W\pi\}$$

and the vanishing ideal is the ideal of all elements of the polynomial algebra $\mathbb{Q}[H_1, \dots, H_n] \simeq \mathfrak{U}^0$ vanishing on this set of points in $\mathbb{A}_{\mathbb{Q}}^n$, where we regard the H_a for $a = 1, \dots, n$ as coordinate functions on $\mathbb{A}_{\mathbb{Q}}^n$. Unlike the quantized case, here we can take H_1, \dots, H_n to be any \mathbb{Q} -basis of $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} Y \subset \mathfrak{g}$; see 5.1. This is a major difference between the generalized Schur algebras and their q -analogues.

In the general (quantized) case, the discrete set of points is

$$\{(v^{\langle H_1, \lambda \rangle}, \dots, v^{\langle H_n, \lambda \rangle}) \mid \lambda \in W\pi\}$$

and the vanishing ideal is the ideal of all elements of the polynomial algebra $\mathbb{Q}[K_{H_1}, \dots, K_{H_n}] \subset \mathbf{U}^0$ vanishing on this set of points in $\mathbb{A}_{\mathbb{Q}(v)}^n$, where we regard the K_{H_a} for $a = 1, \dots, n$ as coordinate functions on $\mathbb{A}_{\mathbb{Q}}^n$. Then the defining ideal of the generalized Schur algebra determined by π is the ideal of \mathfrak{U} (resp., \mathbf{U}) generated by the vanishing ideal.

Note that it immediately follows from these results that the extra defining relations needed to define a generalized Schur algebra as a quotient of \mathbf{U} (resp., \mathfrak{U}) may be found in the zero part of \mathbf{U} (resp., \mathfrak{U}).

The compelling generality and simplicity of this description lies in the wealth of examples that are brought together under its jurisdiction. We easily recover previous results from [5], [2], [6] as special cases of the main result; see §6. The reader is advised to start with the main results, followed by the examples, before tracing through the proofs of the main results, which are somewhat technical.

In order to prove our results, we rely on the presentation (in terms of idempotents in place of Cartan generators) of generalized Schur algebras given in [4]. It was necessary to extend the results of [4] to our slightly more general setup, the details of which are contained in Sections 2 and 3.

One motivation for these results lies in a desire to investigate new instances of Schur-Weyl duality. Let M be a finite dimensional module for \mathbf{U} (or \mathfrak{U}) which admits a weight space decomposition; i.e., an object of \mathcal{C} (see 1.11). Write $\Pi^+(M)$ for the set of dominant weights of M . The set $\pi = \Pi^+(M)$ is necessarily saturated, in the sense of 2.1, so it determines a generalized Schur algebra $\mathbf{S}(\pi)$. We say that the *module* M is saturated if the set of highest weights of its composition factors coincides with $\Pi^+(M)$. If M is a saturated module, then we may identify the generalized Schur algebra $\mathbf{S}(\pi)$ with the image $\rho(\mathbf{U})$ of the representation $\rho : \mathbf{U} \rightarrow \text{End}(M)$ affording the \mathbf{U} -module structure on M . This condition is necessary for the functor $\text{Hom}_{\mathbf{S}(\pi)}(M, -)$ to provide a strong connection between the representation theories of \mathbf{U} and its centralizer algebra $\text{End}_{\mathbf{U}}(M) = \text{End}_{\mathbf{S}(\pi)}(M)$.

In type A if one takes M to be a tensor power of the natural module, then $\mathbf{S}(\pi)$ is the q -Schur algebra and $S(\pi)$ the classical Schur algebra, see 6.1. In type B the tensor powers of the natural module are not generally saturated modules, so in 6.6 we treat the generalized Schur algebra coming from tensor powers of the spin representation, which are in fact always saturated modules (see Appendix A).

It is worth mentioning here an interesting conjecture relating to the question of saturation of tensor powers.

Conjecture. *Let V be a finite dimensional irreducible \mathbf{U} -module which is an object of \mathcal{C} (see 1.11). Then the module $V^{\otimes r}$ is saturated for all $r \geq 0$ if and only if V is minuscule.*

There is some evidence for this conjecture. It is easy to see that in order for $V^{\otimes r}$ to be saturated for all r it is necessary that V be minuscule. Taking $r = 1$ we see that V itself must be saturated, which forces it to be minuscule. Moreover, the converse is known to hold in case V is the natural module in types A , C , D , as well as when V is the spin module in type B . (These are all minuscule.)

1. BASIC NOTATION

Notation introduced here will be used throughout. Our conventions are similar to those of Lusztig's book [8].

1.1. Cartan datum. Let a Cartan datum be given. By definition, a Cartan datum consists of a finite set I and a symmetric bilinear form $(\ , \)$ on the free abelian group $\mathbb{Z}[I]$ taking values in \mathbb{Z} , such that:

- (a) $(i, i) \in \{2, 4, 6, \dots\}$ for any i in I .
- (b) $2(i, j)/(i, i) \in \{0, -1, -2, \dots\}$ for any $i \neq j$ in I .

1.2. Weyl group. The Weyl group W associated to the given Cartan datum is defined as follows. For any $i \neq j$ in I such that $(i, i)(j, j) - (i, j)^2 > 0$ let $h(i, j) \in \{2, 3, 4, \dots\}$ be determined by the equality

$$\cos^2 \frac{\pi}{h(i, j)} = \frac{(i, j)}{(i, i)} \frac{(j, i)}{(j, j)}.$$

One has $h(i, j) = h(j, i) = 2, 3, 4$ or 6 according to whether $\frac{2(i, j)}{(i, i)} \frac{2(j, i)}{(j, j)}$ is $0, 1, 2$, or 3 . For any $i \neq j$ such that $(i, i)(j, j) - (i, j)^2 \leq 0$ one sets $h(i, j) = \infty$.

The braid group is the group given by generators s_i ($i \in I$) satisfying for each $i \neq j$ such that $h(i, j) < \infty$ the relations

$$s_i s_j \cdots = s_j s_i \cdots$$

where on both sides of the equality we alternate between the factors s_i and s_j in the given order until there are $h(i, j)$ factors on each side. The Weyl group W is the group on the same set of generators, satisfying the same relations, along with the relations $s_i^2 = 1$ for all $i \in I$. It is naturally a quotient of the braid group.

1.3. Root datum. A root datum associated to the given Cartan datum consists of two finitely generated free abelian groups X, Y and a perfect bilinear pairing $\langle \ , \ \rangle : Y \times X \rightarrow \mathbb{Z}$ along with embeddings $I \rightarrow Y$ ($i \mapsto h_i$) and $I \rightarrow X$ ($i \mapsto \alpha_i$) such that

$$\langle h_i, \alpha_j \rangle = 2 \frac{(i, j)}{(i, i)}$$

for all i, j in I .

The image of the embedding $I \rightarrow Y$ is the set $\{h_i\}$ of simple coroots and the image of the embedding $I \rightarrow X$ is the set $\{\alpha_i\}$ of simple roots.

1.4. The assumptions on the root datum imply that

- (a) $\langle h_i, \alpha_i \rangle = 2$ for all $i \in I$;
- (b) $\langle h_i, \alpha_j \rangle \in \{0, -1, -2, \dots\}$ for all $i \neq j \in I$.

In other words, the matrix $(\langle h_i, \alpha_j \rangle)$ indexed by $I \times I$ is a symmetrizable generalized Cartan matrix.

For each $i \in I$ we set $d_i = (i, i)/2$ (note that $d_i \in \{1, 2, 3\}$). Then the matrix $(d_i \langle h_i, \alpha_j \rangle)$ indexed by $I \times I$ is symmetric.

1.5. Let v be an indeterminate. Set $v_i = v^{d_i}$ for each $i \in I$. More generally, given any rational function $P \in \mathbb{Q}(v)$ we let P_i denote the rational function obtained from P by replacing each occurrence of v by $v_i = v^{d_i}$.

Set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. For $a \in \mathbb{Z}$, $t \in \mathbb{N}$ we set

$$\begin{bmatrix} a \\ t \end{bmatrix} = \prod_{s=1}^t \frac{v^{a-s+1} - v^{-a+s-1}}{v^s - v^{-s}}.$$

A priori this is an element of $\mathbb{Q}(v)$, but actually it lies in \mathcal{A} (see [8, 1.3.1(d)]). We set

$$[n] = \begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{v^n - v^{-n}}{v - v^{-1}} \quad (n \in \mathbb{Z})$$

and

$$[n]^! = [1] \cdots [n-1] [n] \quad (n \in \mathbb{N}).$$

Then it follows that

$$\begin{bmatrix} a \\ t \end{bmatrix} = \frac{[a]^!}{[t]^! [a-t]^!} \quad \text{for all } 0 \leq t \leq a.$$

1.6. Let \mathbf{U} be the quantized enveloping algebra associated to the given root datum $(X, \{\alpha_i\}, Y, \{h_i\})$. Thus, \mathbf{U} is the associative $\mathbb{Q}(v)$ -algebra with 1 given by the generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad K_h \quad (h \in Y)$$

and satisfying the relations

- (a) $K_0 = 1$, $K_h K_{h'} = K_{h+h'}$ for all $h, h' \in Y$;
- (b) $K_h E_i = v^{\langle h, \alpha_i \rangle} E_i K_h$ for all $i \in I$, $h \in Y$;
- (c) $K_h F_i = v^{-\langle h, \alpha_i \rangle} F_i K_h$ for all $i \in I$, $h \in Y$;
- (d) $E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{v_i - v_i^{-1}}$ for any $i, j \in I$;
- (e) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} E_i^{(s)} E_j E_i^{(s')} = 0$ for all $i \neq j$;
- (f) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} F_i^{(s)} F_j F_i^{(s')} = 0$ for all $i \neq j$.

In (d) above we set $\tilde{K}_i = K_{d_i h_i} = (K_{h_i})^{d_i}$, $\tilde{K}_{-i} = K_{-d_i h_i} = (K_{h_i})^{-d_i}$ and in (e), (f) above we set $E_i^{(s)} = (E_i)^s / ([s]_i^!)$, $F_i^{(s)} = (F_i)^s / ([s]_i^!)$.

The algebra \mathbf{U} admits a Hopf algebra structure, with coproduct

$$\Delta : \mathbf{U} \rightarrow \mathbf{U} \otimes \mathbf{U}$$

given by the algebra homomorphism satisfying

$$\Delta K_h = K_h \otimes K_h;$$

$$\Delta E_i = E_i \otimes \tilde{K}_i^{-1} + 1 \otimes E_i; \quad \Delta F_i = F_i \otimes 1 + \tilde{K}_i \otimes E_i$$

for each $h \in Y$, $i \in I$. Using Δ one defines a \mathbf{U} -module structure on the tensor product of two given \mathbf{U} -modules in the usual manner.

1.7. The classical case. Setting $K_h = v^h$ and letting v tend to 1 (which forces K_h to approach 1), the algebra \mathbf{U} becomes the universal enveloping algebra $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ (over \mathbb{Q}) of the corresponding Kac-Moody Lie algebra \mathfrak{g} generated by $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} Y$ and $\{e_i, f_i : i \in I\}$, with the defining relations

$$(a) [h, h'] = 0;$$

$$(b) [h, e_i] = \langle h, \alpha_i \rangle e_i;$$

$$(c) [h, f_i] = -\langle h, \alpha_i \rangle f_i;$$

$$(d) [e_i, f_j] = \delta_{ij} h_i;$$

$$(e) (\text{ad } e_i)^{1-\langle h_i, \alpha_j \rangle} e_j = 0 = (\text{ad } f_i)^{1-\langle h_i, \alpha_j \rangle} f_j \quad (i \neq j)$$

for all $h, h' \in \mathfrak{h}$ and all $i, j \in I$.

1.8. Triangular decomposition. Let \mathbf{U}^0 be the subalgebra of \mathbf{U} generated by the K_h , $h \in Y$. It is clear that $\mathbf{U}^0 \simeq \mathbb{Q}(v)[Y]$, the group algebra of Y . Denote by \mathbf{U}^+ (respectively, \mathbf{U}^-) the subalgebra of \mathbf{U} generated by the E_i (respectively, the F_i). Then the map

$$\mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+ \rightarrow \mathbf{U}$$

given by $x \otimes K_h \otimes y \rightarrow x K_h y$ is a vector space isomorphism.

1.9. There is a unique action of the Weyl group W on Y such that $s_i(h) = h - \langle h, \alpha_i \rangle h_i$ for all $i \in I$. Similarly, there is a unique action of W on X such that $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$ for all $i \in I$. Then $\langle s_i(h), \lambda \rangle = \langle h, s_i(\lambda) \rangle$ for all $h \in Y$, $\lambda \in X$. Hence for any $w \in W$ we have

$$\langle w(h), \lambda \rangle = \langle h, w^{-1}(\lambda) \rangle$$

for all $h \in Y$, $\lambda \in X$.

1.10. A Cartan datum is of *finite type* if the symmetric matrix $((i, j))$ indexed by $I \times I$ is positive definite. This is equivalent to the requirement that W is a finite group. A Cartan datum that is not of finite type is said to be of *infinite type*.

A root datum is X -regular, respectively, Y -regular if $\{\alpha_i\}$ (respectively, $\{h_i\}$) is linearly independent in X (respectively, Y). We note for later reference that if the underlying Cartan datum is of finite type then the root datum is automatically both X -regular and Y -regular.

In the case where a root datum is X -regular, there is a partial order on X given by: $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in \sum_i \mathbb{N}\alpha_i$. In the case where the root datum is Y -regular, we define $X^+ = \{\lambda \in X \mid \langle h_i, \lambda \rangle \in \mathbb{N}, \text{ all } i\}$, elements of which are known as dominant weights.

1.11. The category \mathcal{C} . Let V be a \mathbf{U} -module. Given any $\lambda \in X$, set

$$V_\lambda = \{m \in V \mid K_h m = v^{\langle h, \lambda \rangle} m, \text{ for all } h \in Y\}.$$

Let \mathcal{C} be the category whose objects are \mathbf{U} -modules V such that $V = \bigoplus_{\lambda \in X} V_\lambda$ (as a $\mathbb{Q}(v)$ vector space). Morphisms in \mathcal{C} are \mathbf{U} -module maps. The subspaces V_λ are called weight spaces of V ; thus the category \mathcal{C} is the category of \mathbf{U} -modules admitting a type **1** weight space decomposition.

Let $\lambda \in X$. The Verma module $M(\lambda)$ is by definition the quotient of \mathbf{U} by the left ideal $\sum_i \mathbf{U}E_i + \sum_\mu \mathbf{U}(K_\mu - v^{\langle \mu, \lambda \rangle})$. This is an object of \mathcal{C} .

Recall that an object M of \mathcal{C} is *integrable* if for any $m \in M$ and any $i \in I$ there is some positive integer N such that $E_i^{(n)} m = F_i^{(n)} m = 0$ for all $n \geq N$.

1.12. The modules $L(\lambda)$. Assume that the root datum is Y -regular. Then to each $\lambda \in X^+$ corresponds an integrable object $L(\lambda)$ of \mathcal{C} , which may be defined as the quotient of $M(\lambda)$ by the submodule generated by all $\theta_i^{\langle h_i, \lambda \rangle + 1}$ for various $i \in I$, where θ_i denotes the image of E_i in $M(\lambda)$. If the root datum is both X -regular and Y -regular, then $L(\lambda)$ is a simple object in the full subcategory of \mathcal{C} whose objects are the integrable \mathbf{U} -modules [8, 6.2.3]. Moreover, $L(\lambda)$ is not isomorphic to $L(\lambda')$ unless $\lambda = \lambda'$ (for λ, λ' in X^+). These properties hold in case the root datum has finite type, since a root datum of finite type is automatically both X -regular and Y -regular.

Assume now that the root datum has finite type. Then $L(\lambda)$ is of finite dimension over $\mathbb{Q}(v)$, and one has the following complete reducibility property (see [8, 6.3.6]): every integrable \mathbf{U} -module is a direct sum of simple \mathbf{U} -modules isomorphic with $L(\lambda)$ for various $\lambda \in X^+$. Since finite-dimensional objects of \mathcal{C} are integrable, they are completely reducible in the above sense.

2. IDEMPOTENT PRESENTATION OF $\mathbf{S}(\pi)$

In this section we generalize the presentation of [4] to any generalized q -Schur algebra corresponding to a given root datum $(X, \{\alpha_i\}, Y, \{h_i\})$ of finite type and a given finite saturated set $\pi \subset X^+$. In particular, this extends the results of [4] to the reductive case.

2.1. Generalized q -Schur algebras. Fix $(X, \{\alpha_i\}, Y, \{h_i\})$, a root datum whose underlying Cartan datum is of finite type. Recall (see 1.10) that in this case the root datum is both X -regular and Y -regular. A given subset π of X^+ is said to be *saturated* (with respect to the partial order \leq defined in 1.10) if $\lambda \in \pi$ and $\lambda' \in X^+$ with $\lambda' \leq \lambda$ imply $\lambda' \in \pi$.

Given a saturated subset π of X^+ we define $\mathcal{C}[\pi]$ to be the full subcategory of \mathcal{C} (see 1.11) whose objects are the \mathbf{U} -modules M in \mathcal{C} such that $M = \bigoplus_{\lambda \in W\pi} M_\lambda$. Every finite dimensional object of $\mathcal{C}[\pi]$ satisfies the property: every simple composition factor of M is isomorphic to some $L(\lambda)$ for $\lambda \in \pi$.

A *generalized q -Schur algebra* is a quotient of the form $\mathbf{S}(\pi) = \mathbf{U}/\Lambda$, where Λ is the ideal of \mathbf{U} consisting of the elements of \mathbf{U} annihilating every object of $\mathcal{C}[\pi]$. The ideal Λ is the *defining ideal* of the generalized q -Schur algebra $\mathbf{S}(\pi)$.

2.2. The algebra $\tilde{\mathbf{S}}(\pi)$. Given a root datum $(X, \{\alpha_i\}, Y, \{h_i\})$ and a finite saturated set $\pi \subset X^+$ we define an algebra $\tilde{\mathbf{S}}(\pi)$ to be the associative algebra with 1 given by the generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad 1_\lambda \quad (\lambda \in W\pi)$$

and the relations

- (a) $1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda, \quad \sum_{\lambda \in W\pi} 1_\lambda = 1;$
- (b) $E_i 1_\lambda = 1_{\lambda + \alpha_i} E_i, \quad 1_\lambda E_i = E_i 1_{\lambda - \alpha_i};$
- (c) $F_i 1_\lambda = 1_{\lambda - \alpha_i} F_i, \quad 1_\lambda F_i = F_i 1_{\lambda + \alpha_i};$
- (d) $E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in W\pi} [\langle h_i, \lambda \rangle]_i 1_\lambda;$
- (e) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} E_i^{(s)} E_j E_i^{(s')} = 0 \text{ for } i \neq j;$
- (f) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} F_i^{(s)} F_j F_i^{(s')} = 0 \text{ for } i \neq j$

for all $i, j \in I$ and all $\lambda, \lambda' \in W\pi$.

For (b), (c) above, one must interpret the symbol $1_{\lambda'}$ as zero whenever $\lambda' \notin W\pi$.

2.3. We claim that $\tilde{\mathbf{S}}(\pi)$ is isomorphic with the generalized q -Schur algebra $\mathbf{S}(\pi) = \mathbf{U}/\Lambda$ corresponding to π . This will eventually be proved in 2.10 ahead, after a series of lemmas to prepare the way. From now on fix π and write $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}(\pi)$.

For any $h \in Y$ define an element $K_h \in \tilde{\mathbf{S}}$ by

$$K_h = \sum_{\lambda \in W\pi} v^{\langle h, \lambda \rangle} 1_\lambda.$$

Then in particular $K_0 = 1$. An easy calculation shows that $K_h K_{h'} = K_{h+h'}$ for all $h, h' \in Y$. Thus the elements K_h ($h \in Y$) satisfy 1.6(a).

2.4. **Lemma.** *Let $\{H_1, \dots, H_n\}$ be a basis of the free abelian group Y , and let $\tilde{\mathbf{S}}^0$ be the subalgebra of $\tilde{\mathbf{S}}$ generated by K_{H_1}, \dots, K_{H_n} . (This is the same as the subalgebra generated by all K_h for $h \in Y$.)*

- (a) $1_\lambda \in \tilde{\mathbf{S}}^0$ for all $\lambda \in W\pi$.
- (b) $K_{-H_1}, \dots, K_{-H_n} \in \tilde{\mathbf{S}}^0$.
- (c) $\{1_\lambda \mid \lambda \in W\pi\}$ is a basis for $\tilde{\mathbf{S}}^0$.

Proof. Let $\{\varepsilon'_1, \dots, \varepsilon'_n\}$ be the basis of X dual to $\{H_1, \dots, H_n\}$, so that $\langle H_a, \varepsilon'_b \rangle = \delta_{ab}$ for $a, b \in \{1, \dots, n\}$. For $\lambda \in X$ write

$$\lambda = \sum_{a=1}^n \lambda_a \varepsilon'_a \quad (\lambda_a \in \mathbb{Z}).$$

We have $\lambda_a = \langle H_a, \lambda \rangle$ for each a .

Let $\lambda \in X$ be given. For each $a = 1, \dots, n$ set $\Gamma(a, \lambda) = \{\lambda' \in W\pi \mid \lambda'_a = \lambda_a\}$. Set

$$J_a^\lambda = \prod_{\substack{\lambda'' \in W\pi \\ \lambda'' \notin \Gamma(a, \lambda)}} (K_{H_a} - v^{\lambda''_a})$$

We have equalities

$$\begin{aligned} J_a^\lambda &= \prod_{\lambda''} \left(\sum_{\lambda' \in W\pi} v^{\lambda'_a} 1_{\lambda'} - v^{\lambda''_a} \sum_{\lambda' \in W\pi} 1_{\lambda'} \right) \\ &= \prod_{\lambda''} \left(\sum_{\lambda' \in W\pi} (v^{\lambda'_a} - v^{\lambda''_a}) 1_{\lambda'} \right) \\ &= \sum_{\lambda' \in W\pi} \prod_{\lambda''} (v^{\lambda'_a} - v^{\lambda''_a}) 1_{\lambda'} \end{aligned}$$

where the products are over all $\lambda'' \in W\pi - \Gamma(a, \lambda)$. The idempotent orthogonality relations 2.2(a) were used to interchange the sum and product. Noting that the product in the sum on the last line above

vanishes for any $\lambda' \in W\pi - \Gamma(a, \lambda)$, we obtain the expression

$$J_a^\lambda = \sum_{\lambda' \in \Gamma(a, \lambda)} \prod_{\lambda''} (v^{\lambda'_a} - v^{\lambda''_a}) 1_{\lambda'}$$

where the product in this sum is a non-zero *constant*, since $\lambda'_a = \lambda_a$ for all $\lambda' \in \Gamma(a, \lambda)$. This proves that J_a^λ is (up to a non-zero scalar) the sum of all idempotents $1_{\lambda'}$ for which $\lambda'_a = \lambda_a$. This property holds for all $a = 1, \dots, n$. It follows that the product $J_1^\lambda \cdots J_n^\lambda$ is, up to a non-zero scalar multiple, equal to 1_λ , since 1_λ is the unique idempotent appearing in each of the sums in the product.

By definition J_a^λ belongs to $\tilde{\mathbf{S}}^0$, so the result of the previous paragraph shows that 1_λ (for any $\lambda \in W\pi$) lies within $\tilde{\mathbf{S}}^0$. This proves part (a).

Part (b) follows from part (a) and the definition of K_{-H_a} .

By definition of the K_{H_a} we see that the subalgebra of $\tilde{\mathbf{S}}$ generated by the 1_λ ($\lambda \in W\pi$) contains the K_{H_a} . By part (a) this subalgebra equals $\tilde{\mathbf{S}}^0$. Part (c) now follows from the fact that the 1_λ ($\lambda \in W\pi$) form a family of orthogonal idempotents. \square

2.5. Lemma. *The algebra $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}(\pi)$ is a homomorphic image of \mathbf{U} , via the homomorphism sending $E_i \rightarrow E_i$, $F_i \rightarrow F_i$, $K_h \rightarrow K_h$.*

Proof. The algebra $\tilde{\mathbf{S}}(\pi)$ is by definition generated by all E_i, F_i ($i \in I$) and K_h ($h \in Y$). We have already observed that the generators of $\tilde{\mathbf{S}}$ satisfy 1.6(a).

We remind the reader of the convention $1_\lambda = 0$ for any $\lambda \in X - W\pi$. For any $h \in Y$ we have

$$\begin{aligned} K_h E_i &= \sum_{\lambda \in X} v^{\langle h, \lambda \rangle} 1_\lambda E_i = \sum_{\lambda \in X} v^{\langle h, \lambda \rangle} E_i 1_{\lambda - \alpha_i} \\ &= \sum_{\lambda \in X} v^{\langle h, \lambda + \alpha_i \rangle} E_i 1_\lambda = v^{\langle h, \alpha_i \rangle} E_i \sum_{\lambda \in X} v^{\langle h, \lambda \rangle} 1_\lambda = v^{\langle h, \alpha_i \rangle} E_i K_h \end{aligned}$$

which proves that the generators of $\tilde{\mathbf{S}}$ satisfy 1.6(b). An entirely similar calculation proves that the generators of $\tilde{\mathbf{S}}$ satisfy 1.6(c).

From 2.2(d) and the definitions we obtain equalities

$$\begin{aligned} E_i F_j - F_j E_i &= \delta_{ij} \sum_{\lambda \in W\pi} [\langle h_i, \lambda \rangle]_i 1_\lambda = \delta_{ij} \sum_{\lambda \in W\pi} \frac{v_i^{\langle h_i, \lambda \rangle} - v_i^{-\langle h_i, \lambda \rangle}}{v_i - v_i^{-1}} 1_\lambda \\ &= \delta_{ij} \frac{(\sum_{\lambda \in W\pi} v_i^{\langle h_i, \lambda \rangle} 1_\lambda) - (\sum_{\lambda \in W\pi} v_i^{-\langle h_i, \lambda \rangle} 1_\lambda)}{v_i - v_i^{-1}}. \end{aligned}$$

But $\sum_{\lambda \in W\pi} v_i^{\langle h_i, \lambda \rangle} 1_\lambda = \sum_{\lambda \in W\pi} v^{d_i \langle h_i, \lambda \rangle} 1_\lambda = (\sum_{\lambda \in W\pi} v^{\langle h_i, \lambda \rangle} 1_\lambda)^{d_i} = K_i^{d_i} = \tilde{K}_i$ and by a similar calculation $\sum_{\lambda \in W\pi} v_i^{-\langle h_i, \lambda \rangle} 1_\lambda = \tilde{K}_{-i}$. Thus the above equalities take the form

$$E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{v_i - v_i^{-1}}$$

proving that the generators of $\tilde{\mathbf{S}}$ satisfy 1.6(d).

Relations 1.6(e), (f) are also satisfied by the generators of $\tilde{\mathbf{S}}$ since those relations are identical with 2.2(e), (f). The lemma is proved. \square

2.6. Lemma. *The algebra $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}(\pi)$ is a finite-dimensional semisimple algebra.*

Proof. First, we note that the generators E_i, F_i for all $i \in I$ are nilpotent elements of $\tilde{\mathbf{S}}$, since by 2.2(c), (d) for N sufficiently large we have $E_i^N 1_\lambda = 0 = F_i^N 1_\lambda$ for all $\lambda \in W\pi$. This implies $E_i^N = 0 = F_i^N$ since $\sum 1_\lambda = 1$.

By the triangular decomposition (see 1.8) we have $\mathbf{U} = \mathbf{U}^- \mathbf{U}^0 \mathbf{U}^+$. It follows that $\tilde{\mathbf{S}}$ has a similar decomposition $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}^- \tilde{\mathbf{S}}^0 \tilde{\mathbf{S}}^+$ where $\tilde{\mathbf{S}}^0, \tilde{\mathbf{S}}^-, \tilde{\mathbf{S}}^+$ are defined as the homomorphic images of $\mathbf{U}^0, \mathbf{U}^-, \mathbf{U}^+$, respectively, under the canonical quotient map $\mathbf{U} \rightarrow \tilde{\mathbf{S}}$.

There is an analogue of the Poincare-Birkhoff-Witt (PBW) theorem for \mathbf{U} . There exist (in addition to the given E_i and F_i) root vectors corresponding to each non simple root in the root system and these additional root vectors can easily be shown to satisfy (in $\tilde{\mathbf{S}}$) a commutation relation analogous to 2.2(c), (d). Thus it follows that all the root vectors are nilpotent elements of $\tilde{\mathbf{S}}$. From the PBW theorem it follows that \mathbf{U}^+ has a basis consisting of products of powers of root vectors, taken in some fixed ordering of the positive roots. The nilpotence of the root vectors thus implies that $\tilde{\mathbf{S}}^+$ is finite-dimensional. A similar argument shows that $\tilde{\mathbf{S}}^-$ is finite-dimensional.

Another approach to the finite-dimensionality of $\tilde{\mathbf{S}}^+, \tilde{\mathbf{S}}^-$ can be obtained from the so-called ‘monomial’ basis of \mathbf{U}^+ (see Lusztig [9] for the simply-laced case and Chari and Xi [1] in general).

It follows from 2.4(c) that $\tilde{\mathbf{S}}^0$ is finite-dimensional. Thus, from the triangular decomposition for $\tilde{\mathbf{S}}$ it follows immediately that $\tilde{\mathbf{S}}$ is finite-dimensional.

The algebra $\tilde{\mathbf{S}}$ is a \mathbf{U} -module via the canonical quotient map $\mathbf{U} \rightarrow \tilde{\mathbf{S}}$. It is known that finite-dimensional \mathbf{U} -modules are completely reducible (i.e., a direct sum of simple \mathbf{U} -modules). Hence, $\tilde{\mathbf{S}}$ is semisimple as a

\mathbf{U} -module, hence semisimple as an $\tilde{\mathbf{S}}$ -module. Thus $\tilde{\mathbf{S}}$ is a semisimple algebra. \square

2.7. *Remark.* Although $\tilde{\mathbf{S}}$ is a direct sum of simple \mathbf{U} -modules by the preceding argument, we do not yet know that it is an object of \mathcal{C} . That is a consequence of the next result.

2.8. **Lemma.** *If M is any finite-dimensional $\tilde{\mathbf{S}}$ -module then M is an object of $\mathcal{C}[\pi]$ and the decomposition $M = \oplus_{\lambda \in W\pi} 1_\lambda M$ is a weight space decomposition of M as a \mathbf{U} -module.*

Proof. View M as a \mathbf{U} -module by means of the canonical quotient map $\mathbf{U} \rightarrow \tilde{\mathbf{S}}$. In $\tilde{\mathbf{S}}$ we have the equality $1 = \sum_{\lambda \in W\pi} 1_\lambda$, which implies that $M = \oplus_{\lambda \in W\pi} 1_\lambda M$. Since

$$1_\lambda K_h = K_h 1_\lambda = v^{\langle h, \lambda \rangle} 1_\lambda \quad (h \in Y)$$

we see immediately that $K_h m = v^{\langle h, \lambda \rangle} m$ for any $m \in 1_\lambda M$. This justifies the inclusion $1_\lambda M \subset M_\lambda$, where

$$M_\lambda = \{m \in M \mid K_h m = v^{\langle h, \lambda \rangle} m, \text{ all } h \in Y\}.$$

On the other hand, for a given $\lambda \in W\pi$, assume that $m \in M_\lambda$, so $K_h m = v^{\langle h, \lambda \rangle} m$ for all $h \in Y$. Multiplying by $1_{\lambda'}$ we obtain

$$1_{\lambda'} K_h m = v^{\langle h, \lambda \rangle} 1_{\lambda'} m$$

and it follows that

$$v^{\langle h, \lambda' \rangle} 1_{\lambda'} m = v^{\langle h, \lambda \rangle} 1_{\lambda'} m \quad (h \in Y, \lambda' \in W\pi).$$

Choosing $h \in Y$ judiciously one concludes that $1_{\lambda'} m = 0$ for any $\lambda' \neq \lambda$, and hence

$$m = 1 \cdot m = \sum_{\lambda' \in W\pi} 1_{\lambda'} m = 1_\lambda m.$$

This proves that $m \in 1_\lambda M$, establishing the reverse inclusion $M_\lambda \subset 1_\lambda M$. Hence $M_\lambda = 1_\lambda M$ for all $\lambda \in W\pi$, and thus $M = \oplus_{\lambda \in W\pi} 1_\lambda M = \oplus_{\lambda \in W\pi} M_\lambda$, so M is an object of $\mathcal{C}[\pi]$, as desired. \square

2.9. **Lemma.** *The set $\{L(\lambda) \mid \lambda \in \pi\}$ is the set of isomorphism classes of simple $\tilde{\mathbf{S}}$ -modules, and $\dim \tilde{\mathbf{S}} = \sum_{\lambda \in \pi} (\dim L(\lambda))^2$.*

Proof. The simple $\tilde{\mathbf{S}}$ -modules are necessarily simple objects of \mathcal{C} . Let $\lambda \in X^+ - \pi$. If $L(\lambda)$ was an $\tilde{\mathbf{S}}$ -module, then by 2.8 $L(\lambda)$ would be a direct sum of the weight spaces $L(\lambda)_{\lambda'}$ as λ' varies over $W\pi$. This is a contradiction since $L(\lambda)_\lambda \neq 0$ and $\lambda \notin W\pi$.

On the other hand, for every $\lambda \in \pi$, $L(\lambda)$ inherits a well-defined $\tilde{\mathbf{S}}$ -module structure from its \mathbf{U} -module structure, just by defining the

action on the generators of $\tilde{\mathbf{S}}$ by the obvious formulas. The first claim is proved.

The second claim follows immediately by standard theory of finite-dimensional algebras. \square

2.10. Theorem. *The algebra $\tilde{\mathbf{S}}(\pi)$ is isomorphic with the generalized Schur algebra determined by the given root datum and the saturated set π . In other words, $\tilde{\mathbf{S}}(\pi) \simeq \mathbf{U}/\Lambda$ where Λ is the ideal consisting of all elements of \mathbf{U} annihilating every object of the category $\mathcal{C}[\pi]$.*

Proof. Let A be the kernel of the canonical quotient map $\mathbf{U} \rightarrow \tilde{\mathbf{S}}$. From 2.9 it is clear that A annihilates every object of $\mathcal{C}[\pi]$. Hence $A \subset \Lambda$ where Λ is the ideal consisting of all $u \in \mathbf{U}$ annihilating every object of $\mathcal{C}[\pi]$, and we have a natural quotient map $\tilde{\mathbf{S}} \simeq \mathbf{U}/A \rightarrow \mathbf{U}/\Lambda$.

It follows that \mathbf{U}/Λ is a finite-dimensional semisimple algebra. Each $L(\lambda)$ for $\lambda \in \pi$ is a simple \mathbf{U}/Λ -module, so the dimension of \mathbf{U}/Λ is at least $\sum_{\lambda \in \pi} (\dim L(\lambda))^2$. But the existence of the above quotient map $\tilde{\mathbf{S}} \rightarrow \mathbf{U}/\Lambda$ ensures that the dimension of \mathbf{U}/Λ is at most $\sum_{\lambda \in \pi} (\dim L(\lambda))^2$. This proves equality of dimension between $\tilde{\mathbf{S}}$ and \mathbf{U}/Λ , so the quotient map $\tilde{\mathbf{S}} \rightarrow \mathbf{U}/\Lambda$ must be an isomorphism, and the result is proved. \square

The theorem shows that $\mathbf{S}(\pi) \simeq \tilde{\mathbf{S}}(\pi)$; hence the generators and relations of 2.2 provide a presentation of $\mathbf{S}(\pi)$.

2.11. Corollary. *Let $\dot{\mathbf{U}}$ be Lusztig's modified form of \mathbf{U} . Then $\mathbf{S}(\pi)$ is isomorphic with $\dot{\mathbf{U}}/\dot{\mathbf{U}}[X^+ - \pi]$ (notation of [8, 29.2]); hence $\mathbf{S}(\pi)$ inherits a canonical basis from the canonical basis of $\dot{\mathbf{U}}$.*

Proof. The proof is entirely similar to the proof of [4, 4.2, 4.3]. Details are left to the reader. \square

3. IDEMPOTENT PRESENTATION OF $S(\pi)$

This section treats the classical analogue of the presentation given in the previous section. The defining relations in this case are obtained simply by setting $v = 1$ in the defining relations of 2.2.

This generalizes the presentation of [4] in the $v = 1$ case (the classical case) to any generalized Schur algebra corresponding to a given root datum $(X, \{\alpha_i\}, Y, \{h_i\})$ of finite type and a given finite saturated set $\pi \subset X^+$. In particular, this applies to the reductive case.

3.1. Generalized Schur algebras. Fix a root datum $(X, \{\alpha_i\}, Y, \{h_i\})$ of finite type. Let $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ denote the classical universal enveloping algebra corresponding to the given root datum (see 1.7). Let π be a saturated subset of X^+ .

Let \mathcal{C} to be the category of \mathfrak{U} -modules admitting a weight space decomposition, and let $\mathcal{C}[\pi]$ be the full subcategory of \mathcal{C} whose objects are the \mathfrak{U} -modules M for which $M = \bigoplus_{\lambda \in W\pi} M_\lambda$. Every finite dimensional object of $\mathcal{C}[\pi]$ satisfies the property: every simple composition factor of M is isomorphic to some $L(\lambda)$ for $\lambda \in \pi$.

A *generalized Schur algebra* [3] is a quotient of the form $S(\pi) = \mathfrak{U}/\Lambda$, where Λ is the ideal of \mathfrak{U} consisting of the elements of \mathfrak{U} annihilating every object of $\mathcal{C}[\pi]$. The ideal Λ is the *defining ideal* of the generalized Schur algebra $S(\pi)$.

3.2. The algebra $\tilde{S}(\pi)$. Given a root datum $(X, \{\alpha_i\}, Y, \{h_i\})$ and a finite saturated set $\pi \subset X^+$ let $\tilde{S}(\pi)$ be the associative algebra over \mathbb{Q} with 1 given by the generators

$$e_i \quad (i \in I), \quad f_i \quad (i \in I), \quad 1_\lambda \quad (\lambda \in W\pi)$$

and the relations

- (a) $1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda, \quad \sum_{\lambda \in W\pi} 1_\lambda = 1;$
- (b) $e_i 1_\lambda = 1_{\lambda + \alpha_i} e_i, \quad 1_\lambda e_i = e_i 1_{\lambda - \alpha_i};$
- (c) $f_i 1_\lambda = 1_{\lambda - \alpha_i} f_i, \quad 1_\lambda f_i = f_i 1_{\lambda + \alpha_i};$
- (d) $e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda \in W\pi} \langle h_i, \lambda \rangle 1_\lambda;$
- (e) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} e_i^{(s)} e_j e_i^{(s')} = 0 \text{ for } i \neq j;$
- (f) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} f_i^{(s)} f_j f_i^{(s')} = 0 \text{ for } i \neq j$

for all $i, j \in I$ and all $\lambda, \lambda' \in W\pi$.

For (b), (c) above, one must interpret the symbol $1_{\lambda'}$ as zero whenever $\lambda' \notin W\pi$. In (e), (f) one interprets $e_i^{(s)} = e_i^s/(s!)$, $f_i^{(s)} = f_i^s/(s!)$ as ordinary divided powers.

3.3. Theorem. *There is an algebra isomorphism $S(\pi) \rightarrow \tilde{S}(\pi)$. In other words, $\tilde{S}(\pi) \simeq S(\pi) = \mathfrak{U}/\Lambda$ where Λ is the ideal of \mathfrak{U} consisting of all elements u annihilating all objects of $\mathcal{C}[\pi]$.*

One can prove this by following a parallel approach to the line of argument given in the previous section, the arguments being similar, but easier, in this instance. We omit the details.

4. MAIN RESULT

We now formulate and prove the main result of this paper, see 4.13, which reduces the problem of finding relations to define $\mathbf{S}(\pi)$ as a quotient of \mathbf{U} to the problem of finding equations that cut out a set of points corresponding to the finite set $W\pi$ of weights, as an affine variety.

4.1. Recall (see 1.8) that \mathbf{U}^0 is isomorphic with the group algebra $\mathbb{Q}(v)[Y]$ of the free abelian group Y . Fix an arbitrary basis $\{H_1, \dots, H_n\}$ of Y . From 1.6(a) it follows that the elements

$$K_{H_1}, \dots, K_{H_n}, K_{-H_1}, \dots, K_{-H_n}$$

generate \mathbf{U}^0 as an algebra over $\mathbb{Q}(v)$. Moreover, it is easy to check that relations 1.6(b), (c) for general $h \in Y$ are consequences of those same relations imposed on just the $h = H_1, \dots, H_n$. Hence, \mathbf{U} is the associative $\mathbb{Q}(v)$ -algebra with 1 given by the generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad K_{H_1}, \dots, K_{H_n}, K_{-H_1}, \dots, K_{-H_n}$$

and satisfying the relations

- (a) $K_{H_a} K_{H_b} = K_{H_b} K_{H_a}$, $K_{H_a} K_{-H_a} = 1$ for $a, b = 1, \dots, n$;
- (b) $K_{H_a} E_i = v^{\langle H_a, \alpha_i \rangle} E_i K_{H_a}$ for $i \in I$, $a = 1, \dots, n$;
- (c) $K_{H_a} F_i = v^{-\langle H_a, \alpha_i \rangle} F_i K_{H_a}$ for $i \in I$, $a = 1, \dots, n$;
- (d) $E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{v_i - v_i^{-1}}$ for any $i, j \in I$;
- (e) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} E_i^{(s)} E_j E_i^{(s')} = 0$ for all $i \neq j$;
- (f) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} F_i^{(s)} F_j F_i^{(s')} = 0$ for all $i \neq j$.

4.2. We have an isomorphism of \mathbf{U}^0 with the ring of Laurent polynomials

$$\mathbf{U}^0 \simeq \mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}, K_{-H_1}, \dots, K_{-H_n}].$$

The polynomial algebra $\mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}]$ is a subalgebra of \mathbf{U}^0 . We find it convenient to regard $\mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}]$ as the affine algebra of the affine variety $\mathbb{A}_{\mathbb{Q}(v)}^n$, and we will henceforth regard K_{H_a} for $a = 1, \dots, n$ as coordinate functions on $\mathbb{A}_{\mathbb{Q}(v)}^n$.

Any subset $D \subset X$ determines a corresponding set of points

$$P_D = \{(v^{\langle H_1, \lambda \rangle}, \dots, v^{\langle H_n, \lambda \rangle}) \mid \lambda \in D\}$$

in $\mathbb{A}_{\mathbb{Q}(v)}^n$. If D is finite then so is P_D . In that case, P_D may be regarded as an affine variety in $\mathbb{A}_{\mathbb{Q}(v)}^n$. Assuming that D is finite, let $I(P_D)$ be the vanishing ideal of the set P_D ; $I(P_D)$ is an ideal of $\mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}] \subset \mathbf{U}^0$. In particular, we have the ideal $I(P_{W\pi})$.

4.3. Lemma. *The ideal $I(P_{W\pi})$ is contained in the defining ideal $\Lambda = \{u \in \mathbf{U} \mid u \text{ acts as zero on all objects of } \mathcal{C}[\pi]\}$.*

Proof. Let $f(K_{H_1}, \dots, K_{H_n})$ be an element of $\mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}]$ which vanishes on every point of the affine variety $P_{W\pi} \subset \mathbb{A}_{\mathbb{Q}(v)}^n$. Every K_h ($h \in Y$) acts on vectors in the λ weight space M_λ of any object M of \mathcal{C} , for any $\lambda \in X$, as the scalar $v^{(h, \lambda)}$; thus the function $f(K_{H_1}, \dots, K_{H_n})$ acts on such a weight space as the scalar $f(v^{(H_1, \lambda)}, \dots, v^{(H_n, \lambda)})$, which is zero for any $\lambda \in W\pi$. Thus any element of $I(P_{W\pi})$ acts as zero on all the weight spaces of any object of $\mathcal{C}[\pi]$, whence the result. \square

4.4. Let $J = \sum_{a \in I(P_{W\pi})} \mathbf{U}a\mathbf{U}$ be the ideal of \mathbf{U} generated by $I(P_{W\pi})$. By the preceding lemma, $J \subset \Lambda$, so there is a surjective quotient map $\mathbf{U}/J \rightarrow \mathbf{U}/\Lambda = \mathbf{S}(\pi)$. For ease of notation, set $\mathbf{T} = \mathbf{U}/J$. Our task is to show that the quotient map $\mathbf{T} \rightarrow \mathbf{S}(\pi)$ is an isomorphism. This will be accomplished eventually in 4.13 below. The strategy of proof is to produce a surjection $\theta : \mathbf{S}(\pi) \rightarrow \mathbf{T}$ by showing that \mathbf{T} has a set of generators satisfying the defining relations of $\mathbf{S}(\pi)$, given in 2.2.

We begin with an examination of a certain quotient of

$$\mathbf{U}^0 = \mathbb{Q}(v)[K_{\pm H_1}, \dots, K_{\pm H_n}]$$

determined by an arbitrary finite set P of points in $\mathbb{A}_{\mathbb{Q}(v)}^n$.

4.5. Lemma. *Let P be any finite set of points in $\mathbb{A}_{\mathbb{Q}(v)}^n$ such that $K_{H_a}(p) \neq 0$ for all $a = 1, \dots, n$ and all $p \in P$. Let $\phi : \mathbf{U}^0 \rightarrow \bigoplus_{p \in P} \mathbb{Q}(v)$ be the evaluation homomorphism sending $f \in \mathbf{U}^0$ to the vector $(f(p))_{p \in P}$. Then $\ker \phi = I'(P)$ where $I'(P)$ is the ideal of \mathbf{U}^0 generated by $I(P)$.*

Proof. Evidently $I'(P) \subset \ker \phi$, so it is enough to establish the reverse inclusion. Let $f \in \mathbb{Q}(v)[K_{\pm H_1}, \dots, K_{\pm H_n}]$ be a Laurent polynomial such that $f(K_{H_1}(p), \dots, K_{H_n}(p)) = 0$ for all $p \in P$. There exist negative integers m_1, \dots, m_n such that $f = (K_{H_1}^{m_1} \cdots K_{H_n}^{m_n})g$ with $g \in \mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}]$. Then for all $p \in P$ we have

$$\begin{aligned} 0 &= f(K_{H_1}(p), \dots, K_{H_n}(p)) \\ &= (K_{H_1}(p)^{m_1} \cdots K_{H_n}(p)^{m_n})g(K_{H_1}(p), \dots, K_{H_n}(p)). \end{aligned}$$

Since $K_i(p) \neq 0$ for $i = 1, \dots, n$ it follows that $g(K_{H_1}(p), \dots, K_{H_n}(p)) = 0$ for all $p \in P$. Thus $g \in I(P)$ and hence $f \in I'(P)$. \square

4.6. Thus we have an explicit isomorphism $\mathbf{U}^0/I'(P) \simeq \bigoplus_{p \in P} \mathbb{Q}(v)$, for any $P \subset \mathbb{A}_{\mathbb{Q}(v)}^n$ satisfying the condition of the lemma. Let 1_p be an element of $\mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}] \subset \mathbf{U}^0$ satisfying the condition $1_p(q) = \delta_{pq}$ for all $q \in P$. Then the elements $\{1_p \mid p \in P\}$ are mutually orthogonal idempotents and $\sum_p 1_p$ is the identity element of $\mathbf{U}^0/I'(P)$.

The algebra \mathbf{T} admits a triangular decomposition, $\mathbf{T} = \mathbf{T}^- \mathbf{T}^0 \mathbf{T}^+$, where \mathbf{T}^- , \mathbf{T}^0 , \mathbf{T}^+ are defined respectively as the image of \mathbf{U}^- , \mathbf{U}^0 , \mathbf{U}^+ . The canonical quotient map $\mathbf{U} \rightarrow \mathbf{U}/J$ induces a map $\mathbf{U}^0 \rightarrow \mathbf{U}/J$ whose kernel is $\mathbf{U}^0 \cap J$. Thus $\mathbf{T}^0 \simeq \mathbf{U}^0/(\mathbf{U}^0 \cap J)$. Clearly $I'(P_{W\pi}) \subset \mathbf{U}^0 \cap J$. Thus we have a sequence of algebra surjections:

$$\mathbf{U}^0/I'(P_{W\pi}) \rightarrow \mathbf{T}^0 \rightarrow \mathbf{S}^0(\pi).$$

(The definition of $\mathbf{S}^0(\pi)$ appeared in the proof of 2.6.) The vector space dimension of $\mathbf{U}^0/I'(P_{W\pi})$ and $\mathbf{S}^0(\pi)$ are both equal to the cardinality of $W\pi$; hence the above surjections are algebra isomorphisms.

4.7. Now let E_i, F_i, K_h ($i \in I, h \in Y$) denote the images of the respective elements of \mathbf{U} under the quotient map $\mathbf{U} \rightarrow \mathbf{T} = \mathbf{U}/J$. Denote by 1_λ the idempotent 1_{p_λ} corresponding to the point

$$p_\lambda = (v^{\langle H_1, \lambda \rangle}, \dots, v^{\langle H_n, \lambda \rangle}).$$

The $\{1_\lambda \mid \lambda \in W\pi\}$ form a basis for the algebra \mathbf{T}^0 . Recall that we are seeking a surjection $\theta : \mathbf{S}(\pi) \rightarrow \mathbf{T}$. To produce such a surjection, it is enough to show that the elements $E_i, F_i, 1_\lambda$ ($i \in I, \lambda \in W\pi$) satisfy the defining relations 2.2(a)–(f).

4.8. Lemma. *For all $h \in Y$ the identity $K_h = \sum_{\lambda \in W\pi} v^{\langle h, \lambda \rangle} 1_\lambda$ holds in the algebra \mathbf{T}^0 .*

Proof. According to 4.6 in light of the identification $\mathbf{T}^0 \simeq \mathbf{U}^0/I'(P_{W\pi})$, we see that

$$(a) \quad K_{H_a} = \sum_{\lambda \in W\pi} K_{H_a}(p_\lambda) 1_\lambda = \sum_{\lambda \in W\pi} v^{\langle H_a, \lambda \rangle} 1_\lambda,$$

an identity in the algebra \mathbf{T}^0 for $a = 1, \dots, n$. From this it follows that

$$(b) \quad K_{-H_a} = \sum_{\lambda \in W\pi} v^{-\langle H_a, \lambda \rangle} 1_\lambda.$$

Indeed, one easily verifies that the product of the right hand side of (a) with the the right hand side of (b) is equal to 1 in \mathbf{T}^0 . Write

$h = \sum_{a=1}^n z_a H_a$ where $z_a \in \mathbb{Z}$. Then from 1.6(a) it follows that

$$(c) \quad K_h = \prod_{\{a|z_a>0\}} (K_{H_a})^{z_a} \prod_{\{a|z_a<0\}} (K_{-H_a})^{-z_a}$$

where all exponents on the right hand side are positive integers. But for any positive integer t one has

$$K_{\pm H_a}^t = \left(\sum_{\lambda \in W\pi} v^{\langle \pm H_a, \lambda \rangle} 1_\lambda \right)^t = \sum_{\lambda \in W\pi} v^{\langle \pm t H_a, \lambda \rangle} 1_\lambda$$

using the orthogonality of the 1_λ . The result follows from this and (c), once again using the orthogonality of the 1_λ . \square

4.9. Lemma. *The identity $E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in W\pi} [\langle h_i, \lambda \rangle]_i 1_\lambda$ holds in the algebra \mathbf{T} , for any $i, j \in I$.*

Proof. From the defining relation 1.6(d) and the preceding lemma we have

$$\begin{aligned} E_i F_j - F_j E_i &= \delta_{ij} \frac{(\sum_{\lambda \in W\pi} v^{\langle h_i, \lambda \rangle} 1_\lambda)^{d_i} - (\sum_{\lambda \in W\pi} v^{-\langle h_i, \lambda \rangle} 1_\lambda)^{d_i}}{v_i - v_i^{-1}} \\ &= \delta_{ij} \frac{(\sum_{\lambda \in W\pi} v^{d_i \langle h_i, \lambda \rangle} 1_\lambda) - (\sum_{\lambda \in W\pi} v^{-d_i \langle h_i, \lambda \rangle} 1_\lambda)}{v_i - v_i^{-1}} \\ &= \delta_{ij} \sum_{\lambda \in W\pi} \frac{v^{d_i \langle h_i, \lambda \rangle} - v^{-d_i \langle h_i, \lambda \rangle}}{v_i - v_i^{-1}} 1_\lambda \\ &= \delta_{ij} \sum_{\lambda \in W\pi} [\langle h_i, \lambda \rangle]_i 1_\lambda \end{aligned}$$

as desired. \square

4.10. Lemma. *Let $f(K_{H_1}, \dots, K_{H_n})$ be an element of the polynomial ring $\mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}] \subset \mathbf{U}^0$ and let $\lambda \in X$. Regard f as a regular function on $\mathbb{A}_{\mathbb{Q}(v)}^n$. The value of $f(v^{\pm \langle H_1, \alpha_j \rangle} K_{H_1}, \dots, v^{\pm \langle H_n, \alpha_j \rangle} K_{H_n})$ on the point p_λ is the same as the value of $f(K_{H_1}, \dots, K_{H_n})$ on the point $p_{\lambda \pm \alpha_j}$, for any $j \in I$.*

Proof. For any $\lambda \in X$ the value of $f(K_{H_1}, \dots, K_{H_n})$ at the point p_λ is

$$f(v^{\langle H_1, \lambda \rangle}, \dots, v^{\langle H_n, \lambda \rangle}).$$

Similarly, the value of $f(v^{\pm \langle H_1, \alpha_j \rangle} K_{H_1}, \dots, v^{\pm \langle H_n, \alpha_j \rangle} K_{H_n})$ at p_λ is

$$f(v^{\langle H_1, \lambda \pm \alpha_j \rangle}, \dots, v^{\langle H_n, \lambda \pm \alpha_j \rangle}),$$

which is the same as the value of $f(K_{H_1}, \dots, K_{H_n})$ at the point $p_{\lambda \pm \alpha_j}$, as desired. \square

4.11. Lemma. *Suppose $\lambda \in W\pi$. Then in the algebra \mathbf{T}^0*

- (a) $1_\lambda(v^{\pm\langle H_1, \alpha_j \rangle} K_{H_1}, \dots, v^{\pm\langle H_n, \alpha_j \rangle} K_{H_n}) = 1_{\lambda+\alpha_j}$ if $\lambda + \alpha_j \in W\pi$.
- (b) $1_\lambda(v^{\pm\langle H_1, \alpha_j \rangle} K_{H_1}, \dots, v^{\pm\langle H_n, \alpha_j \rangle} K_{H_n}) = 0$ if $\lambda + \alpha_j \notin W\pi$.

Proof. Let $D = W\pi \cup \{\omega \pm \alpha_j \mid \omega \in W\pi, j = 1, \dots, n\}$. Set $Q = \mathbf{U}/I'(P_D)$. Note that \mathbf{T}^0 is a non-unital subalgebra of Q . By Lemma 4.10 we have in the algebra Q

$$\begin{aligned} 1_\lambda(v^{-\langle H_1, \alpha_j \rangle} K_{H_1}, \dots, v^{-\langle H_n, \alpha_j \rangle} K_{H_n})(p_\mu) &= 1_\lambda(p_{\mu-\alpha_j}) \\ &= \delta_{\lambda, \mu-\alpha_j} = \delta_{\lambda+\alpha_j, \mu} \end{aligned}$$

for any $\mu \in X$, $j \in I$. The result now follows from the remarks at the beginning of paragraph 4.6, since $\lambda + \alpha_j \in D$. \square

4.12. Lemma. *Let $\lambda \in W\pi$. In the algebra \mathbf{T} we have*

- (a) $E_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} E_i & \text{if } \lambda + \alpha_i \in W\pi \\ 0 & \text{otherwise.} \end{cases}$
- (b) $F_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} F_i & \text{if } \lambda - \alpha_i \in W\pi \\ 0 & \text{otherwise.} \end{cases}$
- (c) $1_\lambda E_i = \begin{cases} E_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in W\pi \\ 0 & \text{otherwise.} \end{cases}$
- (d) $1_\lambda F_i = \begin{cases} F_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in W\pi \\ 0 & \text{otherwise.} \end{cases}$

Proof. From relation 1.6(b) we have $E_i K_h = v^{-\langle h, \alpha_i \rangle} K_h E_i$ for all $h \in Y$, $i \in I$. For any polynomial $f(K_{H_1}, \dots, K_{H_n})$ it follows that

$$E_i f(K_{H_1}, \dots, K_{H_n}) = f(v^{-\langle H_1, \alpha_i \rangle} K_{H_1}, \dots, v^{-\langle H_n, \alpha_i \rangle} K_{H_n}) E_i.$$

Thus in the algebra \mathbf{T} we have

$$E_i 1_\lambda(K_{H_1}, \dots, K_{H_n}) = 1_\lambda(v^{-\langle H_1, \alpha_i \rangle} K_{H_1}, \dots, v^{-\langle H_n, \alpha_i \rangle} K_{H_n}) E_i$$

which by 4.11 equals $1_{\lambda+\alpha_i}$ if $\lambda + \alpha_i \in W\pi$, and equals 0 if $\lambda + \alpha_i \notin W\pi$. This proves part (a); the argument in the remaining cases is similar. \square

Our main result is the following.

4.13. Theorem. *Let $\{H_1, \dots, H_n\}$ be a \mathbb{Z} -basis of Y . The defining ideal Λ (see 2.1) of the generalized q -Schur algebra $\mathbf{S}(\pi)$ is the two-sided ideal of \mathbf{U} generated by $I(P_{W\pi})$. In particular, Λ is generated by its intersection with \mathbf{U}^0 .*

Proof. The existence of a quotient map $\mathbf{T} \rightarrow \mathbf{S}(\pi)$ in 4.4 shows that $\dim \mathbf{T} \geq \dim \mathbf{S}(\pi)$.

On the other hand, \mathbf{T} is generated by the elements E_i, F_i ($i \in I$) and 1_λ ($\lambda \in W\pi$). The orthogonality of the idempotents 1_λ shows that these generators satisfy 2.2(a), and 4.9 shows that they satisfy 2.2(d). They satisfy 2.2(b), (c) by 4.12 and 2.2(e), (f) are automatic by 1.6(e), (f). It thus follows that \mathbf{T} is a quotient of $\mathbf{S}(\pi)$, and thus $\dim \mathbf{T} \leq \dim \mathbf{S}(\pi)$. This proves that

$$\dim \mathbf{S}(\pi) = \dim \mathbf{T},$$

and thus the quotient map $\mathbf{T} \rightarrow \mathbf{S}(\pi)$ is an isomorphism. The result now follows from the definitions $\mathbf{T} = \mathbf{U}/J$, $\mathbf{S}(\pi) = \mathbf{U}/\Lambda$, and the definition of J given in 4.4. \square

Finally, we wish to obtain a more explicit description of the defining ideal $I(P_{W\pi})$. We continue to work with an arbitrary basis H_1, \dots, H_n of the free abelian group Y . Let L_1, \dots, L_n be the corresponding dual basis of X under the perfect pairing. We continue to view the elements $K_a := K_{H_a}$ ($a = 1, \dots, n$) as coordinate functions on n -dimensional affine space $\mathbb{A}_{\mathbb{Q}(v)}^n$. Any element $\lambda \in X$ determines a point

$$p_\lambda = (v^{\langle H_1, \lambda \rangle}, \dots, v^{\langle H_n, \lambda \rangle}) \in \mathbb{A}_{\mathbb{Q}(v)}^n$$

and we regard elements of \mathbf{U}^0 as well-defined functions on P_X via $K_h(p_\lambda) = v^{\langle h, \lambda \rangle}$, for any $h \in Y$, $\lambda \in X$. In the following, we will sometimes use the convenient shorthand $\lambda_a := \langle H_a, \lambda \rangle$, for $a = 1, \dots, n$.

For $h \in Y$, we write $h = \sum c_a H_a \in Y$ in the form $h = h_+ + h_-$ where h_+ (respectively h_-) is the part of the sum indexed by all a such that $c_a > 0$ (respectively $c_a < 0$). We thus have $K_h = K_{h_+} K_{h_-}$. Let Δ be the ideal of \mathbf{U}^0 generated by all

$$F_h = \prod_{\lambda \in W\pi} (K_h - v^{\langle h, \lambda \rangle}) \quad (h \in Y).$$

Since $K_{h_-} \in \mathbf{U}^0$ is invertible, an alternate set of generators for Δ is given by $G_h = \prod_{\lambda \in W\pi} (K_{h_+} - v^{\langle h, \lambda \rangle} K_{h_-}^{-1})$ (for $h \in Y$). We have $K_{h_-} G_h = F_h$ for each $h \in Y$. The generators G_h lie in the polynomial ring $\mathbb{Q}(v)[K_1, \dots, K_n]$.

4.14. Lemma. *Consider the set $Z(\Delta)$ consisting of all $p_\lambda \in P_X$ such that $F(p_\lambda) = 0$ for all $F \in \Delta$. Then $Z(\Delta) = P_{W\pi}$.*

Proof. It is clear that $P_{W\pi} \subset Z(\Delta)$. Now take $p \in Z(\Delta)$. It is clear that $p = p_s$ for some $s = \sum s_a L_a \in X$. Since every $F_h(p_s) = 0$, it follows that for all $h \in Y$ there exists $\lambda \in W\pi$ with $\langle h, \lambda \rangle = \langle h, s \rangle$. If $h =$

$\sum c_a H_a$, then $\langle h, \lambda \rangle = \langle h, s \rangle$ implies that the point $(c_1, \dots, c_n) \in \mathbb{A}_{\mathbb{Q}}^n$ lies on the codimension one hyperplane \mathcal{S}_λ determined by the equation $\sum (s_a - \lambda_a) H_a = 0$. So if $(c_1, \dots, c_n) \notin \bigcup_{\lambda \in W\pi} \mathcal{S}_\lambda$, then $\langle h, \lambda \rangle \neq \langle h, s \rangle$ for all $\lambda \in W\pi$. But $p_s \in Z(\Delta)$ and so no such point (c_1, \dots, c_n) can exist. Thus we must have $\bigcup_{\lambda \in W\pi} \mathcal{S}_\lambda = \mathbb{A}_{\mathbb{Q}}^n$, which implies that $s = \lambda$ for some $\lambda \in W\pi$. Hence $p_s \in P_{W\pi}$ and the lemma is proved. \square

Let Δ' be the ideal of $\mathbb{Q}(v)[K_1, \dots, K_n]$ generated by all G_h . By the comments above, $Z(\Delta) = Z(\Delta')$ and so by the Lemma we have that $Z(\Delta') = P_{W\pi}$. It will be established next that $\mathbb{Q}(v)[K_1, \dots, K_n]/\Delta'$ is reduced, i.e., has no nilpotent elements. We may work over the algebraic closure \mathbb{F} of $\mathbb{Q}(v)$ since nilpotent elements remain nilpotent under field extension.

Let \mathcal{O}_λ be the local ring of $\mathbb{A}_{\mathbb{F}}^n$ at p_λ . Since $Z(\Delta')$ is finite, we have an isomorphism (see [7, Chapter 2, §9, Prop. 6])

$$\mathbb{F}[K_1, \dots, K_n]/\Delta' \rightarrow \prod_{\lambda \in W\pi} \mathcal{O}_\lambda/\Delta' \mathcal{O}_\lambda.$$

Now if each $\mathcal{O}_\lambda/\Delta' \mathcal{O}_\lambda$ has no nilpotent elements, then the desired result will be attained. This will be the case if each $\mathcal{O}_\lambda/\Delta' \mathcal{O}_\lambda$ is regular since a regular local ring is a domain.

4.15. Lemma. *The local ring $\mathcal{O}_\lambda/\Delta' \mathcal{O}_\lambda$ is regular.*

Proof. Fix $\lambda \in W\pi$ and choose $C = (c_{ab}) \in \mathbf{SL}_n(\mathbb{Z})$ such that for all a, b , $c_{ab} \geq 0$ and for all $\mu \neq \lambda$, $\langle H'_a, \lambda \rangle \neq \langle H'_a, \mu \rangle$ where $H'_a = c_{a1}H_1 + \dots + c_{an}H_n$. (The existence of such a C is proved below.) The condition $c_{ab} \geq 0$ for all a, b implies that $F_{H'_a} = G_{H'_a}$ and the condition that $\langle H'_a, \lambda \rangle \neq \langle H'_a, \mu \rangle$ for $\mu \neq \lambda$ implies that $K_{H'_a}(p_\lambda) \neq K_{H'_a}(p_\mu)$ for all $\lambda \neq \mu$. Hence, the generator $F_{H'_a}$ of Δ' is an invertible multiple of $(K_{H'_a} - v^{\langle H'_a, \lambda \rangle})$ in the local ring \mathcal{O}_λ . A simple calculation shows that

$$\frac{\partial(K_{H'_a} - v^{\langle H'_a, \lambda \rangle})}{\partial K_b}(p_\lambda) = c_{ab} v^{-\lambda_b} v^{\langle H'_a, \lambda \rangle}.$$

Hence the Jacobian matrix for $K_{H'_1} - v^{\langle H'_1, \lambda \rangle}, \dots, K_{H'_n} - v^{\langle H'_n, \lambda \rangle}$ at p_λ is the matrix

$$\mathcal{J} = (c_{ab} v^{-\lambda_b} v^{\langle H'_a, \lambda \rangle})_{1 \leq a, b \leq n}.$$

If for all a and b we multiply column b of \mathcal{J} by v^{λ_b} and we divide row a by $v^{\langle H'_a, \lambda \rangle}$, we obtain the matrix C . Since $C \in \mathbf{SL}_n(\mathbb{Z})$, it follows that $\det(\mathcal{J}) \neq 0$. Therefore $\mathcal{O}_\lambda/\Delta' \mathcal{O}_\lambda$ is a regular local ring.

It remains only to establish the existence of a suitable C with the desired properties. This amounts to choosing a “generic” element of $\mathbf{SL}_n(\mathbb{Z})$ with non-negative entries. Here are the details. Fix $\lambda \in W\pi$.

If $\langle H'_a, \lambda \rangle = \langle H'_a, \mu \rangle$, then $\mathbf{c}(a) := (c_{a1}, \dots, c_{an})$ lies on the hyperplane $\mathcal{S}_\mu = (\lambda_1 - \mu_1)H_1 + \dots + (\lambda_n - \mu_n)H_n = 0$. Thus we need to find a C such that none of its rows lies in the finite union $\bigcup_{\mu \neq \lambda} \mathcal{S}_\mu$.

Let C be any matrix in $\mathbf{SL}_n(\mathbb{Z})$ with non-negative entries. Suppose some $\mathbf{c}(a)$ lies on one of the \mathcal{S}_μ . Pick a row $\mathbf{c}(b) \notin \mathcal{S}_\mu$; this is possible since C has rank n . If $\mathbf{c}(a)$ is not in some of the hyperplanes, choose $m > 0$ so that $\mathbf{c}(a) + m\mathbf{c}(b)$ is also not in those hyperplanes. Then $\mathbf{c}(a) + m\mathbf{c}(b)$ is not in \mathcal{S}_μ . Replace $\mathbf{c}(a)$ by $\mathbf{c}(a) + m\mathbf{c}(b)$; the determinant does not change and the entries of C remain non-negative. Repeating that process, we replace $\mathbf{c}(a)$ with a row that is in none of the hyperplanes. \square

We can now obtain the desired explicit description of $I(P_{W\pi})$.

4.16. Proposition. *$I(P_{W\pi})$ is the ideal of $\mathbb{Q}(v)[K_{H_1}, \dots, K_{H_n}]$ generated by all $\prod_{\lambda \in W\pi} (K_h - v^{\langle h, \lambda \rangle})$ for $h \in Y$.*

Proof. It suffices to show that $I(P_{W\pi}) = \Delta'$. By the remarks following the proof of Lemma 4.14, we have $P_{W\pi} = Z(\Delta')$. By Lemma 4.15 it follows that Δ' is a radical ideal. Hence by the Nullstellensatz we have $I(P_{W\pi}) = I(Z(\Delta')) = \sqrt{\Delta'} = \Delta'$, as desired. \square

4.17. Corollary. *The defining ideal Λ of $\mathbf{S}(\pi)$ is the two sided ideal of \mathbf{U} generated by all $\prod_{\lambda \in W\pi} (K_h - v^{\langle h, \lambda \rangle})$ for $h \in Y$.*

Proof. This is immediate from Theorem 4.13 and the preceding proposition. \square

Although this result gives an infinite set of explicit generators for the ideal Λ , it should be noted that Λ is actually finitely generated (since \mathbf{U} is Noetherian). Picking out a finite set of generators of the above form seems difficult in general, although we have managed to do that in various examples, treated below in Section 6.

5. THE CLASSICAL CASE

There is a $v = 1$ analogue of the main result, Theorem 4.13, and the purpose of this section is to formulate it. This reduces the problem of finding relations to define a classical generalized Schur algebra $S(\pi)$ as a quotient of \mathfrak{U} to the problem of finding equations that cut out a set of points corresponding to the finite set $W\pi$ of weights, as an affine variety.

5.1. As before, we fix an arbitrary basis $\{H_1, \dots, H_n\}$ of the abelian group Y . This is a \mathbb{Q} -basis of $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} Y \subset \mathfrak{g}$. It is easy to check that relations 1.7(a), (b), (c) for general $h \in \mathfrak{h}$ are consequences of those same relations imposed on just the $h = H_1, \dots, H_n$. (In fact, this is true if one takes $\{H_1, \dots, H_n\}$ to be any \mathbb{Q} -basis of $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} Y$.) It follows that \mathfrak{U} is the associative algebra with 1 over \mathbb{Q} generated by the

$$e_i, f_i \quad (i \in I) \quad H_1, \dots, H_n$$

and satisfying the relations

- (a) $[H_a, H_b] = 0$;
- (b) $[H_a, e_i] = \langle H_a, \alpha_i \rangle e_i$;
- (c) $[H_a, f_i] = -\langle H_a, \alpha_i \rangle f_i$;
- (d) $[e_i, f_j] = \delta_{ij} h_i$;
- (e) $(\text{ad } e_i)^{1-\langle h_i, \alpha_j \rangle} e_j = 0 = (\text{ad } f_i)^{1-\langle h_i, \alpha_j \rangle} f_j \quad (i \neq j)$

for all $a, b = 1, \dots, n$ and all $i, j \in I$, where $\{H_1, \dots, H_n\}$ is any \mathbb{Q} -basis of $\mathbb{Q} \otimes_{\mathbb{Z}} Y \subset \mathfrak{g}$.

5.2. By the Poincare-Birkhoff-Witt theorem it follows that

$$\mathfrak{U}^0 \simeq \mathbb{Q}[H_1, \dots, H_n];$$

that is, the zero part of \mathfrak{U} is isomorphic with the polynomial algebra in the H_1, \dots, H_n . We regard $\mathbb{Q}[H_1, \dots, H_n]$ as the affine algebra of the affine variety $\mathbb{A}_{\mathbb{Q}}^n$, and we regard H_a for $a = 1, \dots, n$ as coordinate functions on $\mathbb{A}_{\mathbb{Q}}^n$.

Any subset $D \subset X$ determines a corresponding finite set of points

$$P_D = \{(\langle H_1, \lambda \rangle, \dots, \langle H_n, \lambda \rangle) \mid \lambda \in D\}$$

in $\mathbb{A}_{\mathbb{Q}}^n$. Let $I(P_D)$ be the vanishing ideal of the finite set P_D in $\mathbb{A}_{\mathbb{Q}}^n$. This is an ideal of $\mathbb{Q}[H_1, \dots, H_n] \simeq \mathfrak{U}^0$.

5.3. Theorem. *Let $\{H_1, \dots, H_n\}$ be any \mathbb{Q} -basis of $\mathbb{Q} \otimes_{\mathbb{Z}} Y \subset \mathfrak{g}$. The defining ideal Λ (see 3.1) of the generalized Schur algebra $S(\pi)$ is the two-sided ideal of \mathfrak{U} generated by $I(P_{W\pi})$. In particular, Λ is generated by its intersection with \mathfrak{U}^0 .*

The proof parallels the proof of the quantized case in the preceding section, and the details are left to the reader.

Finally, we also have the analogue of Corollary 4.17, proved by similar arguments as in the quantized case. Again, we leave the details to the reader and simply state the result, which gives a more explicit description of the ideal Λ in the classical situation.

5.4. Corollary. *The defining ideal Λ of $S(\pi)$ is the two sided ideal of \mathfrak{U} generated by all $\prod_{\lambda \in W\pi} (h - \langle h, \lambda \rangle)$ for $h \in Y$.*

6. EXAMPLES

We give here as motivation some specific applications of the main results, Theorems 4.13 and 5.3. Not only can we now treat the presentations from [5] and [6] by a unified approach, we are also able to include a completely new example, namely the generalized Schur algebra coming from the set of dominant weights of a tensor power of the spin representation in type B .

Throughout this section, we write $e_{a,b}$ for the matrix unit $e_{a,b} = (\delta_{ai}\delta_{bj})_{1 \leq i,j \leq n}$. In general, Y may be regarded as a sublattice of a Cartan subalgebra \mathfrak{h} of the Lie algebra \mathfrak{g} and hence X may be identified with a sublattice of \mathfrak{h}^* (see 1.7).

6.1. Type A_{n-1} . Let $\mathfrak{U} = \mathfrak{U}(\mathfrak{gl}_n)$ be the universal enveloping algebra of the general linear Lie algebra \mathfrak{gl}_n . We take $X = Y = \mathbb{Z}^n$ with bases $\{H_1, \dots, H_n\}$ of Y and $\{\varepsilon_1, \dots, \varepsilon_n\}$ of X , with pairing $\langle H_a, \varepsilon_b \rangle = \delta_{ab}$ for all pairs of indices a, b . The root datum is given by $h_i = H_i - H_{i+1}$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for all $i \in I = \{1, \dots, n-1\}$. The Cartan datum is defined by $(i, j) = \delta_{ij}$ for all $i, j \in I$. We may identify H_a with the matrix unit $e_{a,a}$ in \mathfrak{gl}_n , for $a = 1, \dots, n$.

Let π be the set of dominant weights occurring in the r th tensor power $V^{\otimes r}$ of the vector representation V . The corresponding set of points in $\mathbb{A}_{\mathbb{Q}}^n$ is precisely the set of partitions of r into n parts (with 0 allowed), so $W\pi$ corresponds with the set of n -part compositions of r . The vanishing ideal on this discrete set of points is generated by

$$(\sum_a H_a) - r, \quad H_a(H_a - 1) \cdots (H_a - r)$$

for $a = 1, \dots, n$. In the quantized case, this is replaced by

$$(\prod_a K_{H_a}) - v^r, \quad (K_{H_a} - 1)(K_{H_a} - v) \cdots (K_{H_a} - v^r)$$

for $a = 1, \dots, n$. Combining these observations with the presentation of \mathfrak{U} and \mathbf{U} given in 5.1 and 4.1 respectively, we recover the main results of [5], obtaining generators and relations for the classical Schur algebras and their q -analogues.

6.2. Type A_{n-1} . We retain the root datum and notation of 6.1, except now we take π to be the set of dominant weights of the module $V^{\otimes r} \otimes V^{*\otimes s}$. The example treated in 6.1 is just the special case $s = 0$. For general r, s the generalized Schur algebra of type A_{n-1} determined by π is called a rational Schur algebra [2].

The set $W\pi$ in this case corresponds to the set of points in $\mathbb{A}_{\mathbb{Q}}^n$ whose components sum to $r - s$, with each proper partial sum of components lying in the interval $[-s, r]$. (In case $s = 0$ this is just a composition of r into n parts.) The vanishing ideal of this discrete point set is generated by

$$(\sum_a H_a) - r + s, \quad (P + s)(P + s - 1) \cdots (P - r)$$

for each proper partial sum P of H_a ($a = 1, \dots, n$). Combined with the presentation of \mathfrak{U} given in 5.1, this gives a presentation of the rational Schur algebra $S(n; r, s)$ studied in [2]. A different (and simpler) presentation of $S(n; r, s)$ is given in [2, 7.3], but in that presentation the quotient map $\mathfrak{U} \rightarrow S(n; r, s)$ is not the natural one sending generators onto generators.

6.3. Type B_n . In type B_n we take $\mathfrak{U} = \mathfrak{U}(\mathfrak{so}_{2n+1}(\mathbb{Q}))$, where the Lie algebra $\mathfrak{so}_{2n+1}(\mathbb{Q})$ is defined relative to the bilinear form whose matrix (in the standard basis) is

$$\begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The $e_{i,i} - e_{n+i,n+i}$ ($1 \leq i \leq n$) form a basis for the diagonal Cartan subalgebra \mathfrak{h} of \mathfrak{g} .

Take $X' = Y' = \mathbb{A}_{\mathbb{Q}}^n$ with fixed \mathbb{Q} -bases $\{\varepsilon_i : 1 \leq i \leq n\}$ of X' and $\{H_i : 1 \leq i \leq n\}$ of Y' . We define a bilinear pairing $\langle \cdot, \cdot \rangle : Y' \times X' \rightarrow \mathbb{Q}$ by the rule $\langle H_i, \varepsilon_j \rangle = \delta_{ij}$. We take $X = \mathbb{Z}^n \cup ((\frac{1}{2}, \dots, \frac{1}{2}) + \mathbb{Z}^n) \subset X'$, with \mathbb{Z} -basis given by

$$\{\varepsilon_i : 1 \leq i \leq n-1\} \cup \{\tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)\}.$$

The simple roots are given by

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i \leq n-1); \quad \alpha_n = \varepsilon_n$$

and the simple coroots are

$$h_i = H_i - H_{i+1} \quad (1 \leq i \leq n-1); \quad h_n = 2H_n.$$

The fundamental weights in this case are given by

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (1 \leq i \leq n-1), \quad \varpi_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2$$

and the set $\{\varpi_1, \dots, \varpi_n\}$ forms another \mathbb{Z} -basis of X . We take Y to be the \mathbb{Z} -span of the simple coroots h_1, \dots, h_n in Y' ; it is easy to see that the restriction of the pairing $\langle \cdot, \cdot \rangle : Y' \times X' \rightarrow \mathbb{Q}$ gives a perfect pairing $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{Z}$ since $\langle h_i, \varpi_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. Thus the datum $(X, \{\alpha_i\}, Y, \{h_i\})$ is a root datum of type B_n .

The indexing set I is $\{1, \dots, n\}$ and we may identify the H_i with the elements $e_{i,i} - e_{n+i,n+i} \in \mathfrak{g}$, for $i \in I$.

Let π be the set $\Pi^+(V^{\otimes r})$, the set of dominant weights occurring in a weight space decomposition of $V^{\otimes r}$, where V is the vector representation. Then π is a saturated subset of X^+ and the set $W\pi$ may be identified with the set of all signed n -part compositions of $r - j$ for $0 \leq j \leq r$ (see [6, Proposition 1.3.1]). By a signed n -part composition of r we mean a tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ such that $\sum |\lambda_i| = r$. The vanishing ideal of this discrete point set in $\mathbb{A}_{\mathbb{Q}}^n$ is generated by the elements

$$(H_i + r)(H_i + r - 1)(H_i + r - 2) \cdots (H_i - r) \\ (J + r)(J + r - 1)(J + r - 2) \cdots (J - r + 1)(J - r)$$

where $i \in I$ and $J = \pm H_1 \pm H_2 \pm \cdots \pm H_n$ varies over all the 2^n possible sign choices. Combined with the presentation of \mathfrak{U} given in 5.1, these observations recover the presentation of $S(\pi)$ given in [6, Theorem 2.1.1].

We note that the module $V^{\otimes r}$ is *not* in general saturated in type B ; see [6, Remark 1.3.4].

To treat the q -analogue $\mathbf{S}(\pi)$ one would need to first express each H_i in terms of the integral basis h_1, \dots, h_n of Y and then rewrite the above elements in terms of the h_i with denominators cleared. This leads to relations which are rather unpleasantly non uniform, but which are easily quantizable. Another approach is to use the set of relations coming from Corollary 4.17.

6.4. Type D_n . In type D_n we take $\mathfrak{U} = \mathfrak{U}(\mathfrak{so}_{2n}(\mathbb{Q}))$, where the Lie algebra $\mathfrak{so}_{2n}(\mathbb{Q})$ is defined relative to the bilinear form whose matrix (in the standard basis) is

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The $e_{i,i} - e_{n+i,n+i}$ ($1 \leq i \leq n$) form a basis for the diagonal Cartan subalgebra \mathfrak{h} of \mathfrak{g} .

Take X', Y' the same as defined in 6.3, with the same pairing $\langle \cdot, \cdot \rangle : Y' \times X' \rightarrow \mathbb{Q}$ and the fixed \mathbb{Q} -bases $\{\varepsilon_i : 1 \leq i \leq n\}$, $\{H_i : 1 \leq i \leq n\}$ of Y', X' respectively such that $\langle H_i, \varepsilon_j \rangle = \delta_{ij}$.

We take $X = \mathbb{Z}^n \cup ((\frac{1}{2}, \dots, \frac{1}{2}) + \mathbb{Z}^n)$ just the same as in 6.3, with the same \mathbb{Z} -basis $\{\varepsilon_i : 1 \leq i \leq n-1\} \cup \{\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)\}$. The simple roots are in this case given by

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i \leq n-1); \quad \alpha_n = \varepsilon_{n-1} + \varepsilon_n$$

and the simple coroots are given by

$$h_i = H_i - H_{i+1} \quad (1 \leq i \leq n-1); \quad h_n = H_{n-1} + H_n.$$

The fundamental weights are in this case given by

$$\begin{aligned} \varpi_i &= \varepsilon_1 + \cdots + \varepsilon_i \quad (1 \leq i \leq n-2), \\ \varpi_{n-1} &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2, \quad \varpi_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2. \end{aligned}$$

and the set $\{\varpi_1, \dots, \varpi_n\}$ is a \mathbb{Z} -basis of X . We define Y to be the \mathbb{Z} -span of the simple coroots h_1, \dots, h_n . The indexing set I is $\{1, \dots, n\}$ and we may identify the H_i with the elements $e_{i,i} - e_{n+i,n+i} \in \mathfrak{g}$, for $i \in I$. Since $\langle h_i, \varpi_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$ one sees immediately that $(X, \{\alpha_i\}, Y, \{h_i\})$ is a root datum of type D_n .

Let π be the set $\Pi^+(V^{\otimes r})$, the set of dominant weights occurring in a weight space decomposition of $V^{\otimes r}$, where V is the vector representation. Then π is a saturated subset of X^+ and the set $W\pi$ may be identified with the set of all signed n -part compositions of $r - 2j$ for $0 \leq j \leq [r/2]$ (see [6, Proposition 1.3.1]). The vanishing ideal of this discrete point set in $\mathbb{A}_{\mathbb{Q}}^n$ is generated by the elements

$$(J+r)(J+r-2)(J+r-4) \cdots (J-r+2)(J-r)$$

where $i \in I$ and $J = \pm H_1 \pm H_2 \pm \cdots \pm H_n$ varies over all the 2^n possible sign choices. Combined with the presentation of \mathfrak{U} given in 5.1, these observations recover the presentation of $S(\pi)$ given in [6, Theorem 2.3.1].

We note that the module $V^{\otimes r}$ is saturated in type D ; see [6, Proposition 1.3.3].

To treat the q -analogue $\mathbf{S}(\pi)$ one faces precisely the same difficulty as discussed at the end of 6.3. Again one needs to express each H_i in terms of the integral basis h_1, \dots, h_n of Y and then rewrite the above elements in terms of the h_i with denominators cleared. As in type B , another approach would be to appeal to Corollary 4.17.

6.5. Type C_n . In type C_n we take $\mathfrak{U} = \mathfrak{U}(\mathfrak{sp}_{2n}(\mathbb{Q}))$, where the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{Q})$ is defined relative to the bilinear form whose matrix (in the standard basis) is

$$\begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix}.$$

The $e_{i,i} - e_{n+i,n+i}$ ($1 \leq i \leq n$) form a basis for the diagonal Cartan subalgebra \mathfrak{h} of \mathfrak{g} .

We take $Y = \mathbb{Z}^n$ and $X = \mathbb{Z}^n$, and we choose bases $\{H_i : 1 \leq i \leq n\}$ of Y and $\{\varepsilon_i : 1 \leq i \leq n\}$ of X , with pairing given by $\langle H_i, \varepsilon_j \rangle = \delta_{ij}$.

The simple roots are the

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i \leq n-1); \quad \alpha_n = 2\varepsilon_n$$

and the simple coroots are

$$h_i = H_i - H_{i+1} \quad (1 \leq i \leq n-1); \quad h_n = H_n.$$

The indexing set is $I = \{1, \dots, n\}$, and we may identify the H_i with the elements $e_{i,i} - e_{n+i,n+i}$, for $i \in I$.

Let π be the set $\Pi^+(V^{\otimes r})$, the set of dominant weights occurring in a weight space decomposition of $V^{\otimes r}$, where V is the vector representation. Then π is a saturated subset of X^+ and the set $W\pi$ may be identified with the set of all signed n -part compositions of $r - 2j$ for $0 \leq j \leq [r/2]$ (see [6, Proposition 1.3.1]). The vanishing ideal of this discrete point set in $\mathbb{A}_{\mathbb{Q}}^n$ is generated by the elements

$$(J+r)(J+r-2)(J+r-4) \cdots (J-r+2)(J-r)$$

where $J = \pm H_1 \pm H_2 \pm \cdots \pm H_n$ varies over all the 2^n possible sign choices. Combined with the presentation of \mathfrak{U} given in 5.1, these observations recover the presentation of $S(\pi)$ given in [6, Theorem 2.2.1].

We note that the module $V^{\otimes r}$ is saturated in type C ; see [6, Proposition 1.3.3].

In this case it is easy to treat the q -analogue, since the set $\{H_1, \dots, H_n\}$ is a \mathbb{Z} -basis of Y . So the defining ideal Λ of $\mathbf{S}(\pi)$ is generated by the elements

$$(K_J - v^r)(K_J - v^{r-2})(K_J - v^{r-4}) \cdots (K_J - v^{-r+2})(K_J - v^{-r})$$

where $J = \pm H_1 \pm H_2 \pm \cdots \pm H_n$ varies over all the 2^n possible sign choices.

6.6. Type B_n — spin module. We treat an entirely new example in this subsection. Retain the root datum as defined in 6.3, except now we take π to be the set $\Pi^+(S^{\otimes r})$, the set of dominant weights occurring in a weight space decomposition of $S^{\otimes r}$, where S is the spin representation of $\mathfrak{so}_{2n+1}(\mathbb{Q})$. We note that the module $S^{\otimes r}$ is saturated in type B ; see Appendix A.

The set $W\pi$ in case $r = 1$ is the set $\Pi(S)$ of weights of S ; the λ in this set are precisely the elements of X satisfying the condition $\langle H_i, \lambda \rangle = \pm \frac{1}{2}$ for all $i \in I$. (The Weyl group W acts through signed permutations.)

For general r the description of $W\pi$ divides naturally into two cases, depending on the parity of r . If $r = 2m$ is even, then one sees easily by a simple induction that the set $W\pi$ consists of those $\lambda \in X$ such

that $\langle H_i, \lambda \rangle \in \{0, \pm 1, \dots, \pm m\}$ for each $i \in I$. If $r = 2m + 1$ then $W\pi$ consists of those $\lambda \in X$ such that $\langle H_i, \lambda \rangle \in \pm \frac{1}{2} + \{0, \pm 1, \dots, \pm m\}$ for each $i \in I$.

The vanishing ideal of this discrete point set in $\mathbb{A}_{\mathbb{Q}}^n$ is generated by the elements

$$(H_i + \frac{r}{2})(H_i + \frac{r}{2} - 1) \cdots (H_i - \frac{r}{2} + 1)(H_i - \frac{r}{2})$$

where $i \in I$. Combined with the presentation of \mathfrak{U} given in 5.1, this gives the following description. The generalized Schur algebra $S(\pi)$ is the associative algebra with 1 over \mathbb{Q} generated by

$$e_i, f_i, H_i \quad (i = 1, \dots, n)$$

and satisfying the relations

- (a) $H_i H_j = H_j H_i$;
- (b) $H_i e_j - e_j H_i = \langle H_i, \alpha_j \rangle e_j$;
- (c) $H_i f_j - f_j H_i = -\langle H_i, \alpha_j \rangle f_j$;
- (d) $e_i f_j - f_j e_i = \delta_{ij} h_i$;
- (e) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} e_i^{(s)} e_j e_i^{(s')} = 0 \quad (i \neq j)$;
- (f) $\sum_{s+s'=1-\langle h_i, \alpha_j \rangle} (-1)^{s'} f_i^{(s)} f_j f_i^{(s')} = 0 \quad (i \neq j)$;
- (g) $(H_i + \frac{r}{2})(H_i + \frac{r}{2} - 1) \cdots (H_i - \frac{r}{2} + 1)(H_i - \frac{r}{2}) = 0$

for all $i, j \in I = \{1, \dots, n\}$. Here $f_i^{(s)} = f_i^s / (s!)$, $e_i^{(s)} = e_i^s / (s!)$ are the usual divided powers.

Treating the q -analogue $\mathbf{S}(\pi)$ one faces difficulties already discussed in previous examples, and probably the best one can do is to appeal to Corollary 4.17.

APPENDIX A. SPIN REPRESENTATION IN TYPE B

The purpose of this appendix is to prove that the tensor powers of the spin representation in type B are always saturated modules, in the sense defined in the introduction.

Retain the root datum introduced in 6.3. This is the root datum for the simply-connected covering group $\mathbf{Spin}(2n+1)$. The fundamental weights for the system are $\varpi_1 = \varepsilon_1$, $\varpi_2 = \varepsilon_1 + \varepsilon_2, \dots, \varpi_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1}$, $\varpi_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$. A basis for the set of dominant weights for the special orthogonal group $\mathbf{SO}(2n+1)$ is $\varpi_1, \varpi_2, \dots, \varpi_{n-1}, 2\varpi_n$; i.e., the dominant weights for the irreducible representations of $\mathbf{SO}(2n+1)$ are the non-negative integral combinations of those weights.

For each non-negative integral combination $\omega = t_1\varpi_1 + \cdots + t_{n-1}\varpi_{n-1} + t_n 2\varpi_n$, let $|\omega| = \sum_{i=1}^n t_i$.

A.1. Lemma. *For each simple root α_i , $|\omega - \alpha_i| \leq |\omega|$.*

Proof. Suppose that $\omega = m_1\varpi_1 + \cdots + m_{n-1}\varpi_{n-1} + m_n 2\varpi_n$, and that $\omega - \alpha_j = k_1\varpi_1 + \cdots + k_{n-1}\varpi_{n-1} + k_n 2\varpi_n$. We need to show that $\sum_{j=1}^n k_j \leq m = \sum_{j=1}^n m_j$.

We have

$$m = m_1 + \cdots + m_n = \langle h_1, \omega \rangle + \cdots + \langle h_{n-1}, \omega \rangle + \frac{\langle h_n, \omega \rangle}{2}$$

and

$$\begin{aligned} k_1 + \cdots + k_n &= \langle h_1, \omega - \alpha_i \rangle + \cdots + \langle h_{n-1}, \omega - \alpha_i \rangle + \frac{\langle h_n, \omega - \alpha_i \rangle}{2} \\ &= m - \langle h_1, \alpha_i \rangle + \cdots + \langle h_{n-1}, \alpha_i \rangle + \frac{\langle h_n, \alpha_i \rangle}{2}. \end{aligned}$$

Using the Cartan datum for B_n , we see that $\langle h_1, \alpha_1 \rangle + \cdots + \langle h_{n-1}, \alpha_1 \rangle + \frac{\langle h_n, \alpha_1 \rangle}{2} = 1$ and that $\langle h_1, \alpha_i \rangle + \cdots + \langle h_{n-1}, \alpha_i \rangle + \frac{\langle h_n, \alpha_i \rangle}{2} = 0$, for $i > 1$. Hence, $k_1 + \cdots + k_n = m$ or $k_1 + \cdots + k_n = m - 1$. \square

For each non-negative integer m , consider the set W_m of dominant weights (for $\mathrm{SO}(2n+1)$) of the form $\omega = m_1\varpi_1 + \cdots + m_{n-1}\varpi_{n-1} + m_n 2\varpi_n$ with $\sum_{j=1}^n m_j = m$ (and each $m_j \geq 0$). Those are the dominant weights ω for $\mathrm{SO}(2n+1)$ such that $|\omega| = m$.

A.2. Lemma. *For each non-negative integer m , the set of dominant weights $\bigcup_{j=0}^m W_j$ is saturated.*

Proof. Let $\omega \in W_m$. Then $\omega = m_1\varpi_1 + \cdots + m_{n-1}\varpi_{n-1} + m_n 2\varpi_n$, where each $m_i \geq 0$, and $m = \sum_{j=1}^n m_j = |\omega|$. Suppose that $\omega - \sum_{j=1}^n t_j \alpha_j$ is a dominant weight where each t_i is a non-negative integer. We need to show that $\omega - \sum_{j=1}^n t_j \alpha_j$ is an element of some W_k with $0 \leq k \leq m$, i.e., that $0 \leq |\omega - \sum_{j=1}^n t_j \alpha_j| \leq m$. Because $\omega - \sum_{j=1}^n t_j \alpha_j$ is dominant, $0 \leq |\omega - \sum_{j=1}^n t_j \alpha_j|$, and by Lemma A.1 we have $|\omega - \sum_{j=1}^n t_j \alpha_j| \leq m$. \square

Let S be the irreducible spin representation of highest weight ϖ_n , and let V be the vector representation of $\mathrm{SO}(2n+1)$, of highest weight ϖ_1 . Let $S^{\otimes k}$ be the k th tensor power of S and let $\wedge^k V$ be the k th exterior power of V . Note that $S^{\otimes 2}$ may be regarded as a module for $\mathrm{SO}(2n+1)$; i.e., the representation of $\mathrm{Spin}(2n+1)$ on $S \otimes S$ factors through $\mathrm{SO}(2n+1)$.

A.3. Lemma. $S \otimes S$ is the sum $k + V + \wedge^2 V + \cdots + \wedge^{n-1} V + \wedge^n V$ of irreducible modules (for $\mathrm{SO}(2n+1)$) of highest weights $0, \varpi_1, \varpi_2, \dots, \varpi_{n-1}$, and $2\varpi_n$.

Proof. This is well known. \square

A.4. Proposition. The set of highest weights of the irreducible factors of $S^{\otimes 2m}$ is $\bigcup_{j=0}^m W_j$.

Proof. The highest weights ω of the irreducible modules in $S \otimes S$ lie in $W_0 \cup W_1$, by Lemma A.3. We claim that the highest weight ω of an irreducible module in the even tensor power $S^{\otimes 2m}$ lie in $\bigcup_{j=0}^m W_j$. In fact, as we observed just above, that is true for $m = 1$. Assume that it is true for $S^{\otimes (2m-2)}$; i.e., that $\omega \in \bigcup_{j=0}^{m-1} W_j$ for the highest weight ω of each irreducible submodule of $S^{\otimes (2m-2)}$. The irreducible submodules of $S^{\otimes 2m}$ occur as submodules of an irreducible factor M of $S \otimes S$ and an irreducible factor N of $S^{\otimes (2m-2)}$. By the case $m = 1$ and by the inductive hypothesis, if ω equals the highest weight of $M \otimes N$, then $\omega \in \bigcup_{j=0}^m W_j$. Any other dominant weight of $M \otimes N$ has the form $\omega - \sum_{i=1}^n d_i \alpha_i$, for non-negative integers d_i . By Lemma A.2, $\omega - \sum_{i=1}^n d_i \alpha_i$ lies in $\bigcup_{j=0}^m W_j$. Hence, the irreducible submodules of $S^{\otimes 2m}$ have highest weights in $\bigcup_{j=0}^m W_j$.

We show next that all dominant weights ω for $\mathrm{SO}(2n+1)$ which lie in $\bigcup_{j=0}^m W_j$ occur as the highest weights of irreducible submodules of $S^{\otimes 2m}$. Lemma A.2 establishes that result for $m = 1$. Suppose that the result holds for $m - 1$, with $m \geq 2$. Then each weight ω in $\bigcup_{j=0}^{m-1} W_j$ occurs as the highest weight of an irreducible submodule of $S^{\otimes (2m-2)}$, and because $S^{\otimes (2m-2)}$ occurs as a submodule of $S^{\otimes 2m} = S \otimes S \otimes S^{\otimes (2m-2)}$ (since k occurs as a submodule of $S \otimes S$), each weight ω in $\bigcup_{j=0}^{m-1} W_j$ occurs as the highest weight of an irreducible submodule of $S^{\otimes 2m}$ also. Next take any weight ω in W_m . Write ω as either $\delta + \varpi_j$ for some $j < m$ or as $\omega = \delta + 2\varpi_n$, for some dominant weight δ . We have $\delta \in W_{m-1}$, and so δ is the highest weight of some irreducible submodule N of $S^{\otimes (2m-2)}$, and ϖ_j (or $2\varpi_n$ as the case may be) is the highest weight of some irreducible submodule M of $S \otimes S$. Hence, ω is the highest weight of $N \otimes M$ of $S^{\otimes 2m}$, and so it is the highest weight of an irreducible submodule of the submodule $N \otimes M$ of $S^{\otimes 2m}$. \square

For weights of the form $\varpi_n + \omega$, where $\omega = t_1 \varpi_1 + \cdots + t_{n-1} \varpi_{n-1} + t_n 2\varpi_n$, for integers t_i , let $|\varpi_n + \omega|$ equal the value $|\omega|$.

A.5. Lemma. The set of dominant weights of the form $\varpi_n + \omega$, for $\omega \in \bigcup_{j=0}^m W_j$, is a saturated set.

The proof is the same as that of Lemma A.2.

A.6. Proposition. *The set of highest weights of the irreducible factors of $S^{\otimes(2m+1)}$ is the set $\{\varpi_n + \omega : \omega \in \bigcup_{j=0}^m W_j\}$.*

Proof. The module $S^{\otimes(2m+1)} = S \otimes S^{\otimes 2m}$ is the sum of submodules of the form $S \otimes M$ where M is an irreducible module with highest weight in $\bigcup_{j=0}^m W_j$. The highest weight of such a tensor product is the highest weight of an irreducible submodule of $S \otimes M$. Such highest weights range over the set described in Lemma A.5, by the proof of Proposition A.4. To show that that set exhausts the highest weights of irreducible submodules of $S^{\otimes(2m+1)}$, we have that the highest weight of $S \otimes M$ has the form $\varpi_n + \omega$ with $\omega \in \bigcup_{j=0}^m W_j$; hence, the dominant weights in $S \otimes M$ have the form

$$\varpi_n + \omega - \sum_{i=1}^n d_i \alpha_i = \varpi_n + \omega - \sum_{i=1}^n d_i \alpha_i,$$

with $d_i \geq 0$, where the dominant weight $\omega - \sum_{i=1}^n d_i \alpha_i$ is an element of $\bigcup_{j=0}^m W_j$, by Lemma A.2. \square

A.7. Theorem. *In type B_n , any tensor power $S^{\otimes r}$ of the spin module S in type B_n is a saturated module (in the sense defined in the introduction).*

Proof. This follows from the definition of saturated modules, given in the introduction, along with results A.2, A.4, A.5, and A.6. \square

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